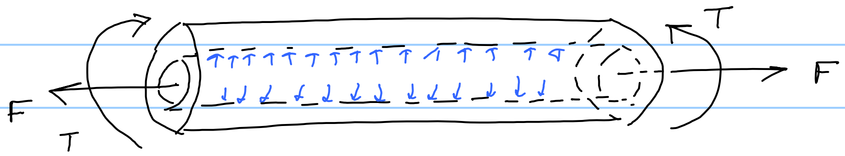
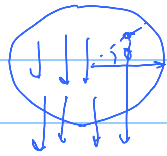
Combined extension- torsion- inflation





$b_r = -\rho g \sin\theta$

$b_\theta = -\rho g \cos\theta$

$u_r(r)$, $u_z(z)$, $u_\theta = \alpha(z)\, r = \cancel{\Omega z}$

extension + inflation

only due to torsion

In this class

stress $\propto$ strain $\propto$ displacement $\Rightarrow$ Balance eqs. are linear in unknown displacement
→ boundary condition is also linear in displacement

⇓ ✗

If eqs are nonlinear then we cannot use superposition!

⇓ Linear elasticity
⇓ we can do superposition
⇓ extension, torsion, inflation can be studied separately and finally combined!

$$[\underline{\underline{\epsilon}}]_{(r,\theta,z)} = \begin{bmatrix} \frac{\partial u_r}{\partial r} = u_r' & \frac{1}{2}\left[\frac{1}{r}\left(\cancel{\frac{\partial u_r}{\partial \theta}} - u_\theta\right) + \frac{\partial u_\theta}{\partial r}\right] = \alpha & \frac{1}{2}\left(\cancel{\frac{\partial u_r}{\partial z}} + \cancel{\frac{\partial u_z}{\partial r}}\right) \\ & \frac{1}{r}\left(\cancel{\frac{\partial u_\theta}{\partial \theta}} + u_r\right) & \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\cancel{\frac{\partial u_z}{\partial \theta}}\right) = \alpha' r \\ & & \frac{\partial u_z}{\partial z} = u_z' \end{bmatrix}$$

$$= \begin{bmatrix} u_r' & 0 & 0 \\ 0 & u_r/r & \frac{\alpha' r}{2} \\ 0 & \frac{\alpha' r}{2} & u_z' \end{bmatrix}$$

$$\sigma_{rr} = \lambda\left(u_r' + \frac{u_r}{r} + u_z'\right) + 2\mu u_r'$$

$$\sigma_{\theta\theta} = \lambda\left(u_r' + \frac{u_r}{r} + u_z'\right) + 2\mu u_r/r$$

$$\sigma_{zz} = \lambda\left(u_r' + \frac{u_r}{r} + u_z'\right) + 2\mu u_z'$$

$$\tau_{r\theta} = 0, \quad \tau_{rz} = 0, \quad \tau_{\theta z} = \mu \alpha' r$$

Stress equilibrium equations.-

$$\frac{\partial \sigma_{rr}}{\partial r} + \cancel{\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}} + \cancel{\frac{\partial \tau_{rz}}{\partial z}} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + b_r = \rho a_r$$

$$\cancel{\frac{\partial \tau_{\theta r}}{\partial r}} + \cancel{\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}} + \frac{\partial \tau_{\theta z}}{\partial z} + \cancel{\frac{2\tau_{r\theta}}{r}} + b_\theta = \rho a_\theta$$

$$\cancel{\frac{\partial \tau_{zr}}{\partial r}} + \cancel{\frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta}} + \frac{\partial \sigma_{zz}}{\partial z} + \cancel{\frac{\tau_{rz}}{r}} + b_z = \rho a_z$$

$$\Downarrow$$

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + \cancel{b_r} = \cancel{\rho a_r}$$

$$\frac{\partial \tau_{\theta z}}{\partial z} + \cancel{b_\theta} = \cancel{\rho a_\theta}$$

$$\frac{\partial \sigma_{zz}}{\partial z} + \cancel{b_z} = \cancel{\rho a_z}$$

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = 0$$

$$\left.\begin{aligned}\frac{\partial \tau_{\theta z}}{\partial z} &= 0\\ \frac{\partial \sigma_{zz}}{\partial z} &= 0\end{aligned}\right\} \Rightarrow \mu\alpha'' r = 0 \Rightarrow \alpha' = \text{constant}$$

$$\Downarrow$$

$$\alpha = \frac{\Omega}{L} z$$

$$\Downarrow$$

$$u_\theta = \frac{\Omega}{L} r z$$

$$\Downarrow$$

$$\frac{\partial}{\partial z}\left[(\lambda+2\mu)u_z' + \lambda\left(u_r' + \frac{u_r}{r}\right)\right] = 0$$

$$\hookrightarrow \underset{\substack{\uparrow\\ z}}{(\lambda+2\mu)u_z'} + \lambda\underbrace{\left(u_r' + \frac{u_r}{r}\right)}_{(r)} = f(r)$$

$$\Downarrow$$

u_z' is a constant!

$u_z' = \epsilon$ or $\underline{\underline{u_z = \epsilon z}}$!!

Radial equation

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = 0$$

$$\sigma_{rr} = (\lambda+2\mu)u_r' + \lambda\frac{u_r}{r} + \lambda\epsilon$$

$$\sigma_{\theta\theta} = (\lambda+2\mu)\frac{u_r}{r} + \lambda u_r' + \lambda\epsilon$$

$$\Rightarrow \sigma_{rr} - \sigma_{\theta\theta} = 2\mu\left(u_r' - \frac{u_r}{r}\right)$$

$$\Rightarrow (\lambda+2\mu)u_r'' + \lambda\frac{u_r'}{r} - \lambda\frac{u_r}{r^2} + \frac{2\mu}{r}\left(u_r' - \frac{u_r}{r}\right) = 0$$

$$\left((\lambda+2\mu)u_r' + \lambda\frac{u_r}{r} + \cancel{\lambda\epsilon}\right)' + \frac{2\mu}{r}\left(u_r' - \frac{u_r}{r}\right) = 0$$

$$\Rightarrow (\lambda+2\mu)\left(u_r' + \frac{u_r}{r}\right)' = 0 \qquad 2\mu\left(\frac{u_r}{r}\right)'$$

$$\Downarrow$$

$$\left(u_r' + \frac{u_r}{r}\right)' = 0 \Rightarrow u_r' + \frac{u_r}{r} = C \Rightarrow \epsilon_{rr} + \epsilon_{\theta\theta} = C$$

$$\left(\frac{1}{r}\left(u_r r\right)'\right)' = 0 \Rightarrow \frac{1}{r}\left(u_r r\right)' = C$$

$$\Rightarrow \left(u_r r\right)' = C r$$

$$\Rightarrow u_r r = C r^2/2 + D$$

$$\Rightarrow \boxed{u_r = C r/2 + D/r}$$