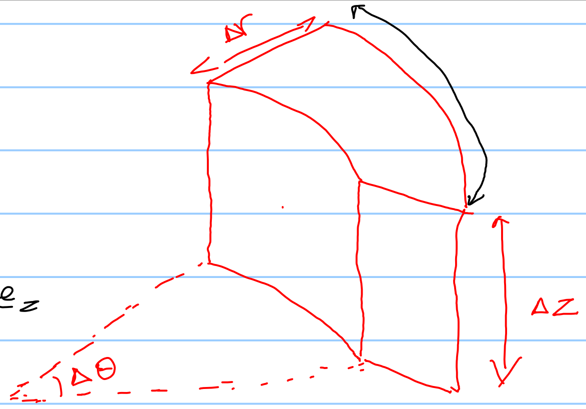
Lec 22 (Stress equilibrium eq. in cylindrical coordinate system)

Note Title 9/23/2022

Traction force on $+r$ and $-r$ planes



$$\underset{\sim}{t}^{+r}\left(r+\frac{\Delta r}{2},\theta,z\right) = \sigma_{rr}\left(r+\frac{\Delta r}{2},\theta,z\right) \underset{\sim}{e}_r + \tau_{\theta r}\left(r+\frac{\Delta r}{2},\theta,z\right) \underset{\sim}{e}_\theta + \tau_{zr}\left(r+\frac{\Delta r}{2},\theta,z\right) \underset{\sim}{e}_z$$

$$\underset{\sim}{t}^{r}\left(r-\frac{\Delta r}{2},\theta,z\right) = -\sigma_{rr}\left(r-\frac{\Delta r}{2},\theta,z\right) \underset{\sim}{e}_r - \tau_{\theta r}\left(r-\frac{\Delta r}{2},\theta,z\right) \underset{\sim}{e}_\theta - \tau_{zr}\left(r-\frac{\Delta r}{2},\theta,z\right) \underset{\sim}{e}_z$$

$$A^{+r} = \left(r+\frac{\Delta r}{2}\right)\Delta\theta\Delta z, \qquad A^{-r} = \left(r-\frac{\Delta r}{2}\right)\Delta\theta\Delta z$$

$$\underset{\sim}{F}^{+r} + \underset{\sim}{F}^{-r} = \underset{\sim}{t}^{+r}\cdot A^{+r} + \underset{\sim}{t}^{-r} A^{-r}$$

$$= \Delta\theta\Delta z\left[\left(r+\frac{\Delta r}{2}\right)\left\{\sigma_{rr}\left(r+\frac{\Delta r}{2},\theta,z\right)\underset{\sim}{e}_r + \cdots\cdots\right\} + \left(r-\frac{\Delta r}{2}\right)\left\{-\sigma_{rr}\left(r-\frac{\Delta r}{2},\theta,z\right)\underset{\sim}{e}_r - \cdots\cdots\right\}\right]$$

$$f = r\sigma_{rr} \qquad = \Delta\theta\Delta z\left[\frac{\partial}{\partial r}(r\sigma_{rr})\underset{\sim}{e}_r\,\Delta r + \frac{\partial}{\partial r}(r\tau_{\theta r})\underset{\sim}{e}_\theta\,\Delta r + \frac{\partial}{\partial r}(r\tau_{zr})\underset{\sim}{e}_z\,\Delta r\right]$$

$$= \underbrace{r\Delta r\Delta\theta\Delta z}\left[\frac{1}{r}\left[r\frac{\partial\sigma_{rr}}{\partial r}+\sigma_{rr}\right]\underset{\sim}{e}_r + \frac{1}{r}\left\{r\frac{\partial\tau_{\theta r}}{\partial r}+\tau_{\theta r}\right\}\underset{\sim}{e}_\theta + \frac{1}{r}\left\{r\frac{\partial\tau_{zr}}{\partial r}+\tau_{zr}\right\}\underset{\sim}{e}_z\right]$$

$$= \Delta V\left[\frac{\partial\sigma_{rr}}{\partial r}\underset{\sim}{e}_r + \frac{\partial\tau_{\theta r}}{\partial r}\underset{\sim}{e}_\theta + \frac{\partial\tau_{zr}}{\partial r}\underset{\sim}{e}_z + \frac{\sigma_{rr}}{r}\underset{\sim}{e}_r + \frac{\tau_{\theta r}}{r}\underset{\sim}{e}_\theta + \frac{\tau_{zr}}{r}\underset{\sim}{e}_z\right]$$

Total traction force:

$$\left[\frac{\partial\sigma_{zz}}{\partial z}\underset{\sim}{e}_z + \frac{\partial\tau_{\theta z}}{\partial z}\underset{\sim}{e}_\theta + \frac{\partial\tau_{rz}}{\partial z}\underset{\sim}{e}_r + \frac{1}{r}\left(\frac{\partial\tau_{r\theta}}{\partial\theta}-\sigma_{\theta\theta}\right)\underset{\sim}{e}_r + \frac{1}{r}\left(\tau_{r\theta}+\frac{\partial\sigma_{\theta\theta}}{\partial\theta}\right)\underset{\sim}{e}_\theta + \frac{\partial\tau_{z\theta}}{\partial\theta}\underset{\sim}{e}_z + \left(\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}}{r}\right)\underset{\sim}{e}_r + \left(\frac{\partial\tau_{\theta r}}{\partial r}+\frac{\tau_{\theta r}}{r}\right)\underset{\sim}{e}_\theta + \left(\frac{\partial\tau_{zr}}{\partial r}+\frac{\tau_{zr}}{r}\right)\underset{\sim}{e}_z\right]\Delta V$$

Body force: $\left(b_r\underset{\sim}{e}_r + b_\theta\underset{\sim}{e}_\theta + b_z\underset{\sim}{e}_z\right)\Delta V$

$$\frac{d}{dt}(\underset{\sim}{p}) = \rho\left(a_r\underset{\sim}{e}_r + a_\theta\underset{\sim}{e}_\theta + a_z\underset{\sim}{e}_z\right)\Delta V$$

$$\nabla\cdot\underset{\sim}{\sigma}$$

$$\frac{\partial}{\partial r}(\underset{\sim}{\sigma})\otimes\underset{\sim}{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}(\underset{\sim}{\sigma})\otimes\underset{\sim}{e}_\theta + \frac{\partial}{\partial z}(\underset{\sim}{\sigma})\otimes\underset{\sim}{e}_z$$

$e_r:\quad \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + b_r = \rho a_r$

$e_\theta:\quad \frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{\tau_{r\theta}+\tau_{\theta r}}{r} + b_\theta = \rho a_\theta$

$e_z:\quad \frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\tau_{zr}}{r} + b_z = \rho a_z$

$$\underline{\underline{\epsilon}} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)$$

$$\underline{u} = u_r \underline{e}_r + u_\theta \underline{e}_\theta + u_z \underline{e}_z$$

$$\nabla \underline{u} = \frac{\partial}{\partial r}(\underline{u}) \otimes \underline{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}(\underline{u}) \otimes \underline{e}_\theta + \frac{\partial}{\partial z}(\underline{u}) \otimes \underline{e}_z$$

$$\left[\frac{\partial u_r}{\partial r}\underline{e}_r + \frac{\partial u_\theta}{\partial r}\underline{e}_\theta + \frac{\partial u_z}{\partial r}\underline{e}_z\right] \otimes \underline{e}_r$$

$$\left[\frac{\partial u_r}{\partial z}\underline{e}_r + \frac{\partial u_\theta}{\partial z}\underline{e}_\theta + \frac{\partial u_z}{\partial z}\underline{e}_z\right] \otimes \underline{e}_z$$

$$\frac{1}{r}\left[\frac{\partial u_r}{\partial \theta}\underline{e}_r + \frac{\partial u_\theta}{\partial \theta}\underline{e}_\theta + \frac{\partial u_z}{\partial \theta}\underline{e}_z + u_r \underline{e}_\theta - u_\theta \underline{e}_r\right] \otimes \underline{e}_\theta$$

$$[\nabla \underline{u}]_{(\underline{e}_r - \underline{e}_\theta - \underline{e}_z)} = \begin{array}{c|c} & \begin{array}{ccc} r & \theta & z \end{array} \\ \begin{array}{c} r \\ \theta \\ z \end{array} & \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) & \frac{\partial u_r}{\partial z} \\ \frac{\partial u_\theta}{\partial r} & \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) & \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial r} & \frac{\partial u_z}{\partial \theta} & \frac{\partial u_z}{\partial z} \end{bmatrix} \end{array}$$

ϵ_{rr} $\quad$ $\epsilon_{r\theta}$

$$[\underline{\underline{\epsilon}}]_{(\underline{e}_r - \underline{e}_\theta - \underline{e}_z)} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2}\left[\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right)\right] & \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\ & \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) & \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \\ & & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

$$\gamma_{r\theta} = 2\left(\underline{\underline{\epsilon}}\,\underline{e}_r\right)\cdot \underline{e}_\theta = 2\epsilon_{r\theta}$$

$$\epsilon_{\theta\theta} = \cancel{2}\left(\underline{\underline{\epsilon}}\,\underline{e}_\theta\right)\cdot \underline{e}_\theta = \epsilon_{\theta\theta}$$