Lecture 2 (Math Prelims)

Note Title 8/5/2022

$$\vec{v} = \sum_{i=1}^{3} (\vec{v}\cdot \underline{e}_i)\, \underline{e}_i = \sum_{i=1}^{3} (\vec{v}\cdot \hat{\underline{e}}_i)\, \hat{\underline{e}}_i$$

$$= v_i\, \underline{e}_i \qquad \hat{v}_i\, \hat{\underline{e}}_i$$

$$\Downarrow$$

$$[\vec{v}]_{(\underline{e}_1-\underline{e}_2-\underline{e}_3)} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \neq \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \end{bmatrix} = [\vec{v}]_{(\hat{\underline{e}}_1-\hat{\underline{e}}_2-\hat{\underline{e}}_3)}$$

A vector and its representation are not the same!

$\underline{v}$: a vector , v : a scalar ! $|v|$

Dot product (scalar product)

$$\underline{a}\cdot\underline{b} = \|\underline{a}\|\, \|\underline{b}\| \cos\theta$$

magnitude (norm)

$$\underline{a}\cdot\underline{b} = \sum_i a_i b_i$$

$$[\underline{a}]_{(\underline{e}_1-\underline{e}_2-\underline{e}_3)} \cdot [\underline{b}]_{(\underline{e}_1-\underline{e}_2-\underline{e}_3)} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= [\underline{a}]^T_{()}\, [\underline{b}]_{()}$$

$$[\underline{a}]_{(\hat{\underline{e}}_1-\hat{\underline{e}}_2-\hat{\underline{e}}_3)} \cdot [\underline{b}]_{(\hat{\underline{e}}_1-\hat{\underline{e}}_2-\hat{\underline{e}}_3)} = \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix} \cdot \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = \begin{bmatrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 \end{bmatrix} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix}$$

Cross-product (Vector product)

$$\underline{a}\times\underline{b} = \|\underline{a}\|\, \|\underline{b}\| \sin\theta\ \underline{c}$$

a unit vector $\perp$ to both $\underline{a}$ & $\underline{b}$ (given by right hand thumb rule)

$$[\underline{a}]_{(\underline{e}_1-\underline{e}_2-\underline{e}_3)} \times [\underline{b}]_{(\underline{e}_1-\underline{e}_2-\underline{e}_3)} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - b_2 a_3 \\ b_1 a_3 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

a skew symmetric matrix

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \underset{\text{axial}([\underline{a}])}{\overset{\text{skew}([\underline{a}])}{\rightleftarrows}} \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Tensor product

$$\underset{\sim}{a} \otimes \underset{\sim}{b} = \underset{\approx}{C}$$

2nd order tensor

→ A tensor is a generalized concept of vector. Just like vectors, tensors are also independent of coordinate system but their mathematical representation depends on the coordinate system.

- → vectors are first order tensors
- → scalars are also independent of coord system and are zeroth order tensors!

$$\underset{\sim}{a} \otimes \underset{\sim}{b}$$

$$[\underset{\sim}{a}]_{()} \otimes [\underset{\sim}{b}]_{()} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{3\times1} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}_{1\times3}$$

$$= \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

→ Matrices are mathematical representation of 2nd order tensors!

$$\underset{\sim}{a} \otimes \underset{\sim}{b} = \sum_i a_i \underset{\sim}{e}_i \otimes \sum_j b_j \underset{\sim}{e}_j = \sum_i \sum_j \underbrace{a_i b_j}_{\text{component}} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$$

$$\underset{\sim}{v} = \sum_i v_i \underset{\sim}{e}_i$$

→ there are 9 basis tensor $(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)$

$$[\underset{\sim}{e}_i]_{()} \otimes [\underset{\sim}{e}_j]_{()}$$

$$[\underset{\sim}{e}_1]_{(\underset{\sim}{e}_1 - \underset{\sim}{e}_2 - \underset{\sim}{e}_3)} \otimes [\underset{\sim}{e}_2]_{(\underset{\sim}{e}_1 - \underset{\sim}{e}_2 - \underset{\sim}{e}_3)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A general 2nd order tensor:

$$\underset{\approx}{C} = \sum_i \sum_j C_{ij} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$$

$$\left[\underline{\underline{C}}\right]_{(\underline{e}_1-\underline{e}_2-\underline{e}_3)} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$\underline{\underline{C}} = \sum_i \sum_j C_{ij} \, \underline{e}_i \otimes \underline{e}_j$$

$$= \sum_i \sum_j \hat{C}_{ij} \, \hat{\underline{e}}_i \otimes \hat{\underline{e}}_j$$

$$C_{ij} \neq \hat{C}_{ij}$$