# Lec 19 (Stress-strain relation)



$$\sigma_{mn} = \frac{1}{2}\left[\sum_i\sum_j C_{ijmn}\,\epsilon_{ij} + \sum_k\sum_l C_{mnkl}\,\epsilon_{kl}\right]$$

$k \quad l$

$$= \frac{1}{2}\left[\sum_k\sum_l C_{klmn}\,\epsilon_{kl} + \sum_k\sum_l C_{mnkl}\,\epsilon_{kl}\right]$$

$$\sigma_{mn} = \sum_k\sum_l \underbrace{\left[\frac{C_{klmn} + C_{mnkl}}{2}\right]}_{}\epsilon_{kl}$$

$$\tilde{C}_{mnkl} = \tilde{C}_{klmn}$$

Major Symmetry

$$\sigma_{mn} = K_{\underset{6}{mn}\,\underset{6}{kl}}\,\epsilon_{kl}$$

$$81 \xrightarrow{\text{Minor Sym}} 36 \xrightarrow{\text{Major sym}} 21$$

→ Most general anisotropic materials require 21 constants!

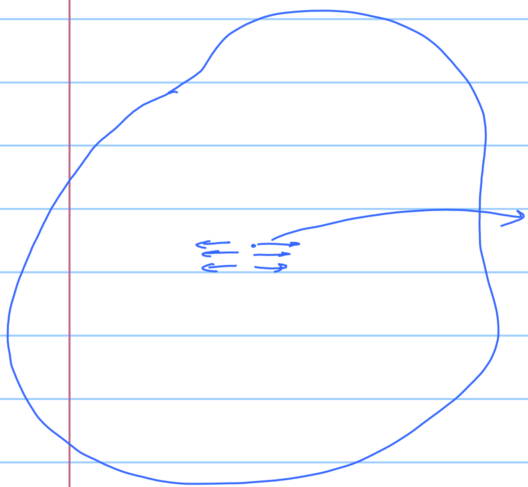
## Voigt Notation

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$[\underline{\sigma}] = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \text{6x6 matrix} \\ \text{of } K_{ijkl} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = [\epsilon]$$

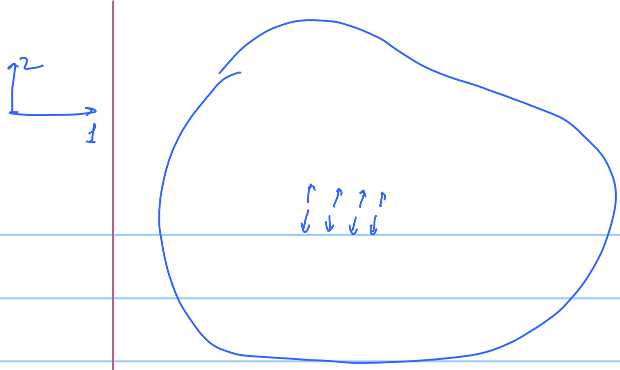
stress vector

symmetric matrix!



$$[\epsilon] = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_{11} = \underline{C_{1111}}\,\epsilon_{11}$$



$$[\underline{\underline{\epsilon}}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_{22} = \underline{C_{2222}} \; \epsilon_{22}$$

$\Rightarrow$ If material properties are same, response should also be same

$\Rightarrow C_{1111} = C_{2222} = C_{3333}$

→ Isotropic materials need only 2 independent constants!

they have same material property in different directions!

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\epsilon}} \qquad (\hat{e}_1 - \hat{e}_2 - \hat{e}_3)$$

$\downarrow (e_1 - e_2 - e_3)$

$$\sigma_{ij} = \sum_{k}\sum_{l} C_{ijkl}\, \epsilon_{kl} \qquad \underline{\hat{\sigma}_{ij} = \sum_{k}\sum_{l} \hat{C}_{ijkl}\, \hat{\epsilon}_{kl}}$$

For isotropic material, $\hat{C}_{ijkl} = C_{ijkl}$ for all coordinate system!

$$\boxed{\sigma_{ij} = \lambda\, tr(\underline{\underline{\epsilon}})\, \delta_{ij} + 2\mu\, \epsilon_{ij}}$$

for isotropic materials

$(\lambda, \mu)$ are the two independent material constants!

→ (Lame's constants)

$$\sigma_{11} = \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) + 2\mu\,\epsilon_{11} = (\lambda + 2\mu)\,\epsilon_{11} + \lambda(\epsilon_{22} + \epsilon_{33})$$

$C_{1111}$ $\quad C_{1122} = C_{1133}$

$$\sigma_{22} = (\lambda + 2\mu)\,\epsilon_{22} + \lambda(\epsilon_{11} + \epsilon_{33})$$

$C_{2222}$ $\quad C_{2211}$

$$\sigma_{33} = (\lambda + 2\mu)\,\epsilon_{33} + \lambda(\epsilon_{11} + \epsilon_{22})$$

$C_{3333}$

$$\sigma_{12} = 2\mu\,\epsilon_{12}, \quad \sigma_{13} = 2\mu\,\epsilon_{13}, \quad \sigma_{23} = 2\mu\,\epsilon_{23}$$

$C_{1313}$ $\quad C_{2323}$

$$\epsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right)$$

$$\epsilon_{22} = \frac{1}{E}\left(\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})\right)$$

$$\epsilon_{33} = \frac{1}{E}\left(\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\right)$$

$$\gamma_{12} = \frac{1}{G}\tau_{12}, \quad \gamma_{13} = \frac{1}{G}\tau_{13}, \quad \gamma_{23} = \frac{1}{G}\tau_{23}$$

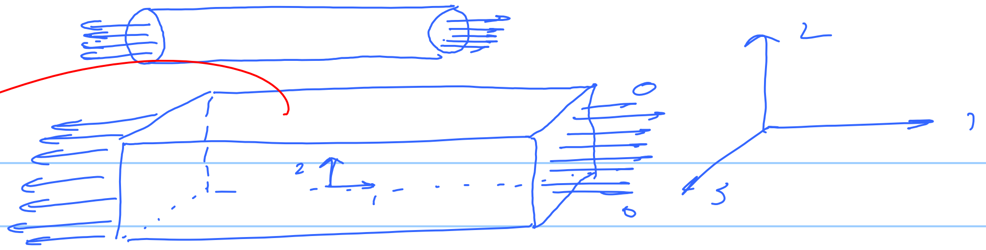
Young's modulus of Elasticity

Poisson's ratio

Shear modulus of Elasticity

$$G = \frac{E}{2(1+\nu)}$$

Uniaxial loading (nothing being applied on lateral surface)




$\gamma_{13}, \gamma_{23}, \gamma_{12}$ are expected to be zero!, $\epsilon_{11}, \epsilon_{22}, \epsilon_{33}$ should be non-zero!

$\sigma_{11}, \sigma_{22}, \sigma_{33}$ should be non-zero!

$\tau_{13}, \tau_{23}, \tau_{12}$ are also expected to be zero!

$$[\underline{\sigma}\,\underline{e}_2] = \begin{bmatrix}\tau_{12}\\ \sigma_{22}\\ \tau_{32}\end{bmatrix}$$

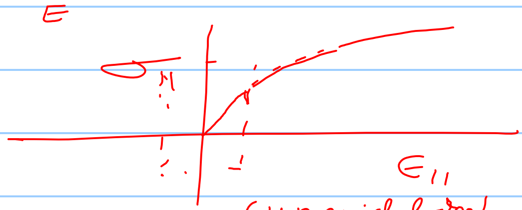
$\Rightarrow$ $\sigma_{22}$ is zero on top and bottom surface

$\Rightarrow$ $\sigma_{33}$ " " on front and back surface

$\Rightarrow$ $\sigma_{22}, \sigma_{33}$ is zero everywhere!

$$\epsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu(\cancel{\sigma_{22}} + \cancel{\sigma_{33}})\right) = \frac{\sigma_{11}}{E}$$

$$\Rightarrow \boxed{\sigma_{11} = E\,\epsilon_{11}}$$




slope of the uniaxial tension test graph at $\epsilon_{11} = 0$!

$$\epsilon_{22} = \frac{1}{E}\left(\cancel{\sigma_{22}} - \nu(\sigma_{11} + \cancel{\sigma_{33}})\right)$$

$$\Rightarrow \epsilon_{22} = -\nu\frac{\sigma_{11}}{E} = -\nu\,\epsilon_{11} \quad \text{(for uniaxial test)}$$

$$\nu = -\frac{\epsilon_{22}}{\epsilon_{11}} \quad \text{(for uniaxial loading)}$$

$$= -\frac{\epsilon_{33}}{\epsilon_{11}}$$