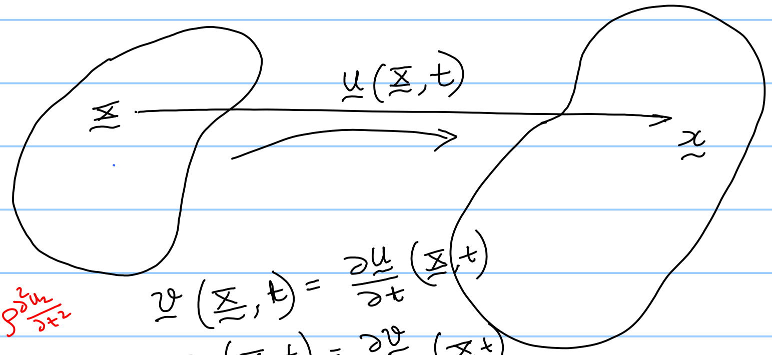
Lecture 18 — Stress-strain relation

Note Title — 9/14/2022

$$\underset{\sim}{\nabla} \cdot \underset{=}{\sigma} + \rho \underset{\sim}{b} = \rho \underset{\sim}{a}$$

$b \to$ force/vol.

force/mass



$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} + \rho b_1 = \rho a_1 \quad \left(= \rho \frac{\partial^2 u_1}{\partial t^2}\right)$$

$$\frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \tau_{23}}{\partial x_3} + \rho b_2 = \rho a_2 \to \rho \frac{\partial^2 u_2}{\partial t^2}$$

$$\frac{\partial \tau_{31}}{\partial x_1} + \frac{\partial \tau_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho b_3 = \rho a_3 \to \rho \frac{\partial^2 u_3}{\partial t^2}$$

$$\underset{\sim}{v}(\underset{\sim}{X}, t) = \frac{\partial \underset{\sim}{u}}{\partial t}(\underset{\sim}{X}, t)$$

$$\underset{\sim}{a}(\underset{\sim}{X}, t) = \frac{\partial \underset{\sim}{v}}{\partial t}(\underset{\sim}{X}, t) = \frac{\partial^2 \underset{\sim}{u}}{\partial t^2}(\underset{\sim}{X}, t)$$

$\underset{=}{\sigma} = \underset{=}{\sigma}^T \Rightarrow$ 6 stress components + 3 acc. comp.

In fluid mechanics, $\underset{\sim}{v}^E(\underset{\sim}{x}, t)$

$$\left.\frac{\partial \underset{\sim}{v}^L(\underset{\sim}{X}, t)}{\partial t}\right|_{\underset{\sim}{X}} =$$

$$\underset{\sim}{a}^E(\underset{\sim}{x}, t) = \left.\frac{\partial \underset{\sim}{v}^E}{\partial t}\right|_{\underset{\sim}{X}} = \left.\frac{\partial \underset{\sim}{v}^E}{\partial \underset{\sim}{x}}\right|_{\underset{\sim}{x}} \left.\frac{\partial \underset{\sim}{x}}{\partial t}\right|_{\underset{\sim}{X}} + \left.\frac{\partial \underset{\sim}{v}^E}{\partial t}\right|_{\underset{\sim}{x}}$$

$$= \underbrace{\underset{\sim}{\nabla}_{\underset{\sim}{x}} \underset{\sim}{v}^E \; \underset{\sim}{v}^E}_{\text{convective acc.}} + \underbrace{\left.\frac{\partial \underset{\sim}{v}^E}{\partial t}\right|_{\underset{\sim}{x}}}_{\text{local acc.}}$$

total derivative

$$\left.\frac{\partial \underset{\sim}{v}^L}{\partial t}\right|_{\underset{\sim}{X}}$$

$$\underset{\sim}{v}^L(\underset{\sim}{X}, t) = \underset{\sim}{v}^E(\underset{\sim}{x}(\underset{\sim}{X}, t), t)$$

→ With stress - strain relation, $\underset{\sim}{\epsilon}$ being a function of $\underset{\sim}{u}$, stress $(\underset{=}{\sigma} - \underset{=}{\epsilon})$ also becomes a function of $\underset{\sim}{u}$

$\Rightarrow$ total unknowns in eq. reduces to 3 (u_1, u_2, u_3)

Stress strain relation

$$\underset{=}{\sigma} = \hat{\underset{=}{\sigma}}(\underset{=}{\epsilon})$$

it can be any function which can only be obtained through experiments!

$$\sigma_{ij} = \hat{\sigma}_{ij}(\epsilon_{kl})$$

$$\hat{\sigma}_{11}(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23} \ldots \epsilon_{21}, \epsilon_{31}, \epsilon_{32})$$

$$\hat{\sigma}_{ij}(\epsilon_{kl}=0) + \sum_{k}\sum_{l} \frac{\partial \hat{\sigma}_{ij}}{\partial \epsilon_{kl}}(\epsilon_{kl}=0) \cdot \epsilon_{kl} + \frac{1}{2!}\frac{\partial^2 \hat{\sigma}_{ij}}{\partial \epsilon_{kl}\partial \epsilon_{mn}}\epsilon_{kl}\epsilon_{mn} + \ldots$$

0

0

stress in the reference configuration

→ usually, ref. config is also stress free!

$$\|\nabla \underline{u}\| \ll 1 \Rightarrow \|\underline{\underline{\epsilon}}\| \ll 1$$

$$\sigma_{ij} = \sum_{k}\sum_{l} \frac{\partial \hat{\sigma}_{ij}}{\partial \epsilon_{kl}}(\epsilon_{kl}=0)\ \epsilon_{kl}$$

$$= \sum_{kl}\sum \underbrace{C_{ijkl}}_{\text{fourth order tensor!}} \epsilon_{kl}$$

81 components present

Due to symmetry of $\underline{\underline{\sigma}}$ $\Rightarrow$ $C_{ijkl} = C_{jikl}$

$$\sigma_{ij} = \sum_{kl}\sum C_{ijkl}\,\epsilon_{kl} = \sum_{k}\sum_{l} \frac{C_{ijkl}+C_{ijlk}}{2}\epsilon_{kl} + \sum_{k}\sum_{l}\frac{C_{ijkl}-C_{ijlk}}{2}\epsilon_{kl}$$

0

$$\frac{1}{2}\sum_{k}\sum_{l} C_{ijkl}\,\epsilon_{kl} - \frac{1}{2}\sum_{k}\sum_{l} C_{ijlk}\,\epsilon_{kl}$$

$$-\frac{1}{2}\sum_{k}\sum_{l} C_{ijlk}\,\epsilon_{lk}$$

$$\hat{C}_{ijkl} = \frac{1}{2}(C_{ijkl}+C_{ijlk})$$

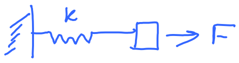
$$\sigma_{ij} = \sum_{k}\sum_{l} \hat{C}_{ijkl}\,\epsilon_{kl}$$

sym in (i,j) and also (k,l)

Minor symmetry of C_{ijkl}

$\Rightarrow$ $81 \rightarrow 36$

Major symmetry



$$E = \frac{1}{2}kx^2 = \frac{1}{2}(kx)\,x = \frac{1}{2}Fx$$

$$\frac{\partial E}{\partial x} = kx = F$$

Energy density stored in the body

$$E = \sum_i\sum_j \frac{1}{2}\sigma_{ij}\,\epsilon_{ij}$$

$$= \frac{1}{2}\sum_i\sum_j\sum_k\sum_l C_{ijkl}\,\epsilon_{kl}\,\epsilon_{ij}$$

$$\sigma_{mn} = \frac{\partial E}{\partial \epsilon_{mn}} = \frac{1}{2}\sum\sum\sum\sum C_{ijkl}\left(\frac{\partial \epsilon_{kl}}{\partial \epsilon_{mn}}\epsilon_{ij} + \epsilon_{kl}\frac{\partial \epsilon_{ij}}{\partial \epsilon_{mn}}\right)$$

$$= \frac{1}{2}\sum\sum\sum\sum C_{ijkl}\left(\delta_{km}\delta_{ln}\,\epsilon_{ij} + \epsilon_{kl}\,\delta_{im}\delta_{jn}\right)$$

$$= \frac{1}{2}\sum_i\sum_j C_{ijmn}\,\epsilon_{ij} + \frac{1}{2}\sum_k\sum_l C_{mnkl}\,\epsilon_{kl}$$