Note Title — 9/13/2022

Lec 17 Strain.

Infinitesimal strain tensor!

$$\underline{\underline{\epsilon}} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)$$

$$\epsilon_{nn} = \left(\underline{\underline{\epsilon}}\,\underline{n}\right)\cdot \underline{n}$$

$$\gamma_{mn} = 2\left(\underline{\underline{\epsilon}}\,\underline{n}\right)\cdot \underline{m}$$

$$\epsilon_v = tr\left(\underline{\underline{\epsilon}}\right)$$

$$\|\nabla \underline{u}\| << 1$$

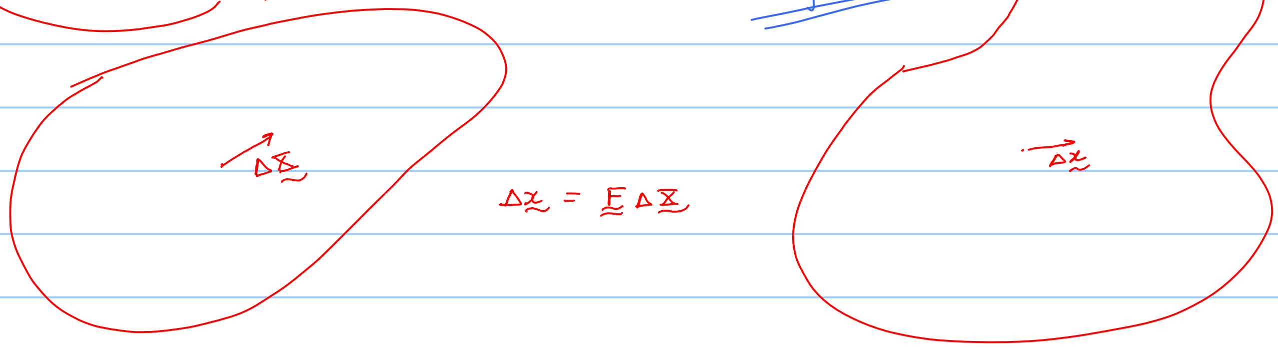
$$\underline{\underline{F}} = \underline{\underline{I}} + \nabla \underline{u}$$

$$= \underline{\underline{I}} + \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right) + \frac{1}{2}\left(\nabla \underline{u} - \nabla \underline{u}^T\right)$$

$$= \underline{\underline{I}} + \underline{\underline{\epsilon}} + \underline{\underline{W}}$$

?

infinitesimal rotation



$$\Delta \underline{x} = \underline{\underline{F}}\,\Delta \underline{X}$$

Rodgrigue's formula for rotation :-

$$\underline{\underline{R}}\left(\underline{a}, \theta\right) = \underline{\underline{I}}\cos\theta + \underline{\underline{a}}\sin\theta + \underline{a}\otimes\underline{a}\left(1-\cos\theta\right)$$

$$\underline{\underline{R}}\left(\underline{a} = \underline{e}_3, \theta\right) = \underline{\underline{I}}\cos\theta + \underline{\underline{e}}_3 \sin\theta + \underline{e}_3 \otimes \underline{e}_3 \left(1-\cos\theta\right)$$

$$[\underline{e}_3] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\cos\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (1-\cos\theta)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\theta << 1$$

$$\cos\theta = 1 - \theta^2/2 + \theta^4/4! \ldots \approx 1$$

$$\sin\theta = \theta - \theta^3/3! + \ldots \approx \theta$$

$$\underline{\underline{I}} \times 1 + \underline{\underline{a}}\,\theta + \underline{a}\otimes\underline{a}\,(1-1)$$

$$\Downarrow$$

$$\underline{\underline{R}}\left(\underline{a}, \theta << 1\right) = \boxed{\underline{\underline{I}} + \theta\, \underline{\underline{a}}}$$

$$\underline{a} \xrightarrow{\text{skew sym. tensor}} \underline{\underline{a}}, \qquad \underline{\underline{a}} \xrightarrow{\text{axial vector}} \underline{a}$$

Infinitesimal rotations are commutative!

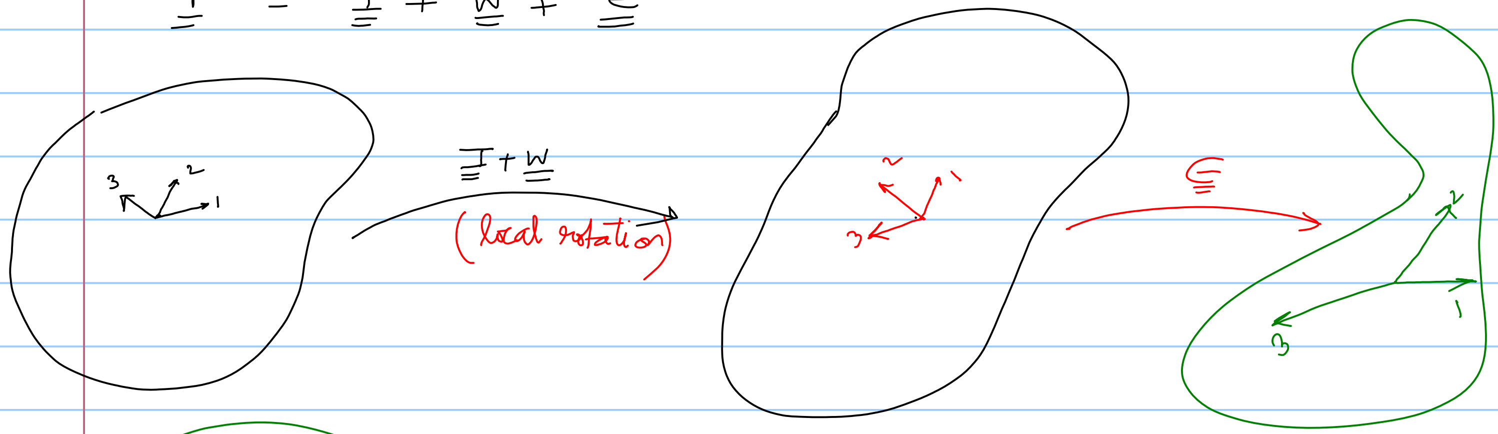
$$[\underline{\underline{W}}] = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ -ve & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ -ve & -ve & 0 \end{bmatrix}$$

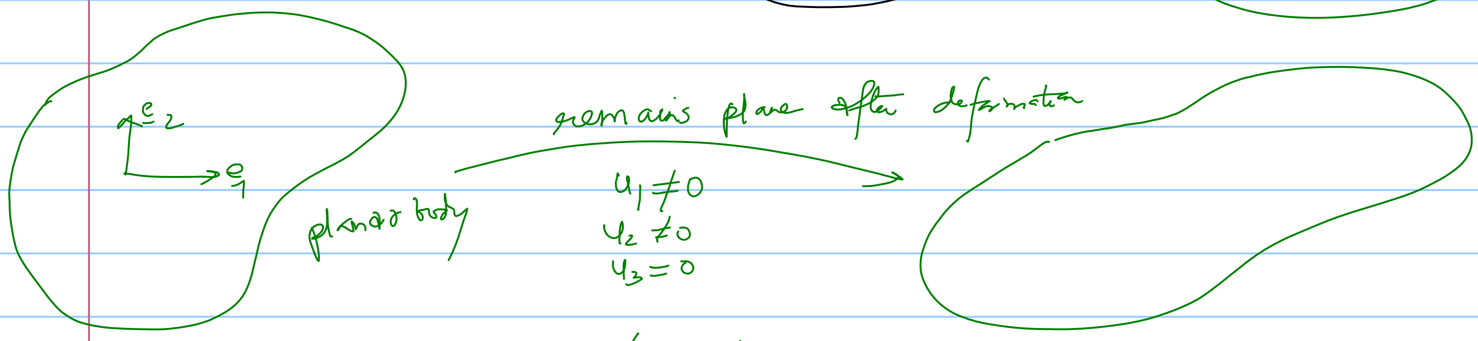
$$[\underline{w}] = \begin{bmatrix} \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \end{bmatrix}$$

$-ve$ $+ve$ $-ve$

$$\theta = \|\underline{w}\| , \qquad \underline{a} = \frac{\underline{w}}{\|\underline{w}\|}$$

$$\underline{\underline{F}} = \overbrace{\underline{\underline{I}} + \underline{\underline{W}}}^{\underline{\underline{R}}} + \underline{\underline{\epsilon}}$$





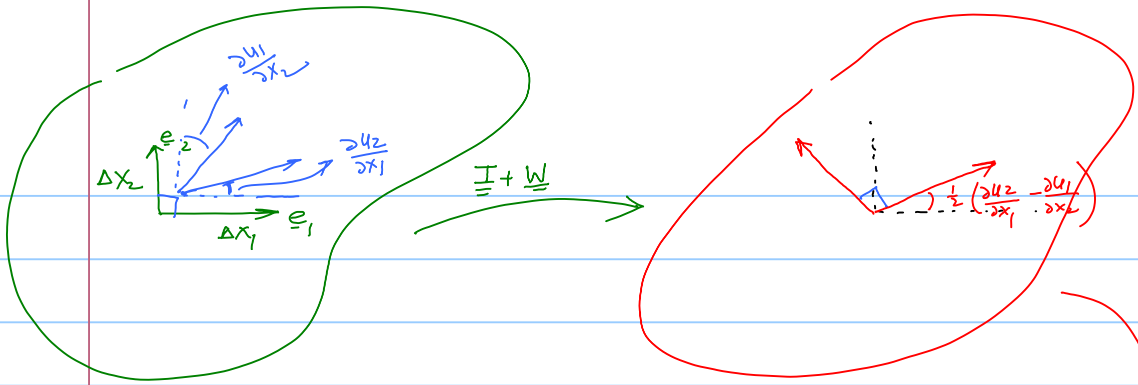
$u_1(x_1, x_2)$

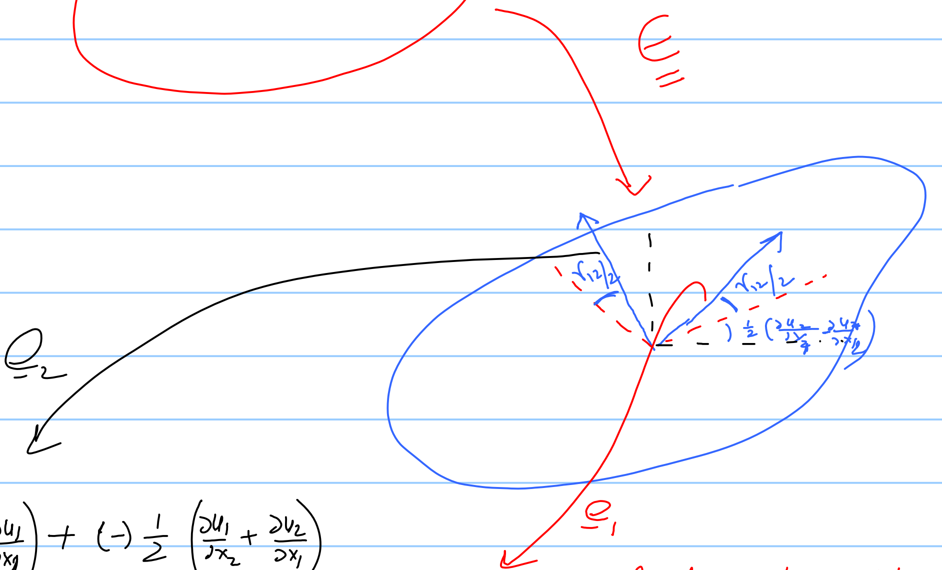
$u_2(x_1, x_2)$

$u_3 = 0.$

$$[\underline{\underline{\epsilon}}] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & 0 \\ & \frac{\partial u_2}{\partial x_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\underline{\underline{W}}] = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & 0 \\ -ve & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\underline{w}] = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \end{bmatrix}$$





total rotation

$$= \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) + (-)\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$= -\frac{\partial u_1}{\partial x_2}$$

Total rotation = local rotation + rotation due to shear

$$= \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) + \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$= \frac{\partial u_2}{\partial x_1}$$

Strain compatibility condition

$$\underline{\underline{\epsilon}} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right), \qquad \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

→ 6 strain functions are getting generated just by 3 displacement functions!

⇒ 6 strain functions cannot be chosen arbitrarily! ✓

$$[\underline{\underline{\epsilon}}] = \epsilon_{xx},\ \epsilon_{yy},\ \epsilon_{zz},\ \epsilon_{xy},\ \epsilon_{yz},\ \epsilon_{zx}$$

$$\epsilon_{xx} \to \frac{\partial u_x}{\partial x}, \quad \epsilon_{yy} \to \frac{\partial u_y}{\partial y}, \quad \epsilon_{xy} \to \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^3 u_x}{\partial x\, \partial y^2} + \frac{\partial^3 u_y}{\partial x^2 \partial y}$$

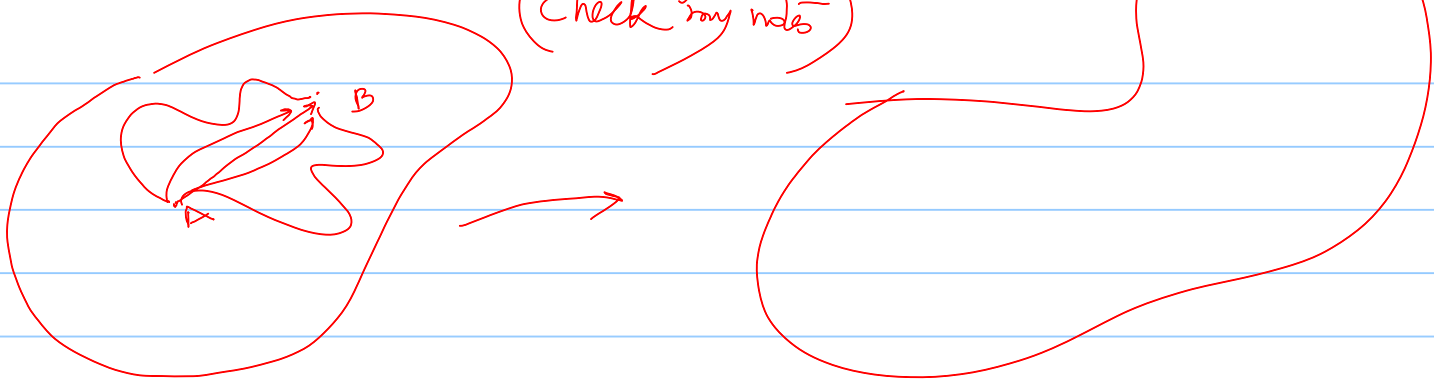
$$= \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_x}{\partial y}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_y}{\partial x}\right)$$

$$= \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}$$

$$\boxed{\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}}$$

→ There are 6 compatibility equation!

(Check your notes)



Strain compatibility conditions ensure that integration of strain is path independent!