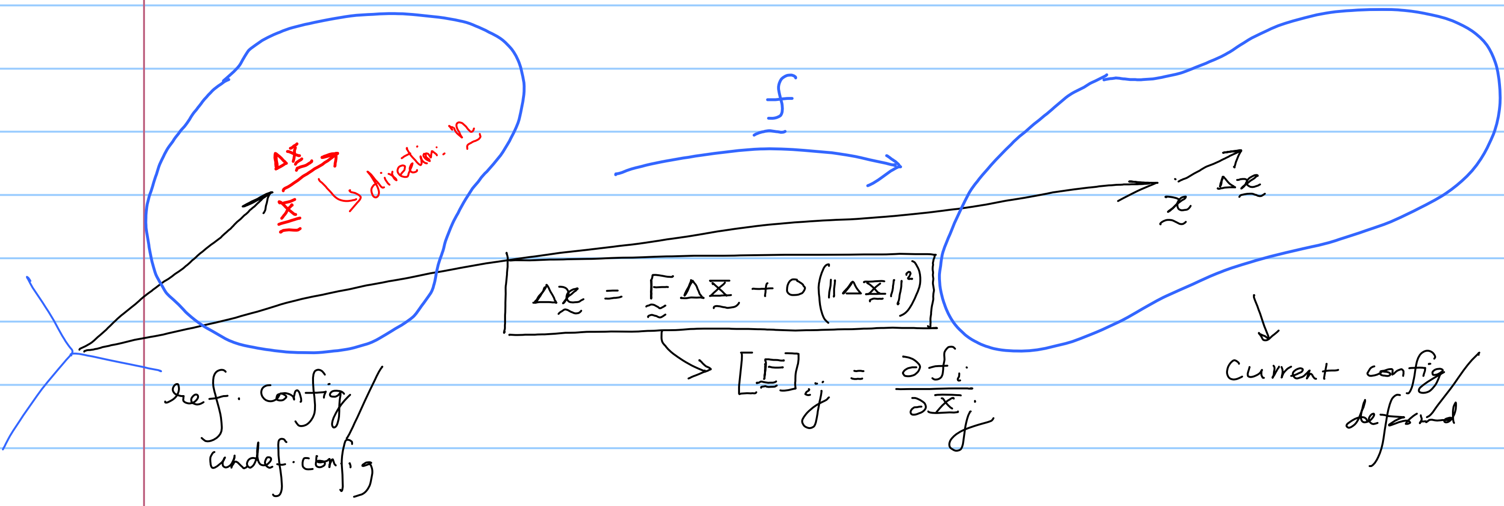
Lecture 15 Strain

Note Title 9/7/2022



$$\boxed{\Delta \underset{\sim}{x} = \underset{\approx}{F}\,\Delta \underset{\sim}{X} + O\left(\|\Delta X\|^2\right)}$$

$$[\underset{\approx}{F}]_{ij} = \frac{\partial f_i}{\partial X_j}$$

λ : stretch in the line element

$$\lambda^2 = \frac{\|\Delta \underset{\sim}{x}\|^2}{\|\Delta \underset{\sim}{X}\|^2} = \frac{\Delta \underset{\sim}{x}}{\|\Delta \underset{\sim}{X}\|} \cdot \frac{\Delta \underset{\sim}{x}}{\|\Delta \underset{\sim}{X}\|} = \frac{\left[\underset{\approx}{F}\,\Delta \underset{\sim}{X} + O\left(\|\Delta X\|^2\right)\right]}{\|\Delta \underset{\sim}{X}\|} \cdot \frac{[\;\cdots\;]}{\|\Delta \underset{\sim}{X}\|}$$

$$= \left[\underset{\approx}{F}\,\underset{\sim}{n} + O\left(\|\Delta X\|\right)\right] \cdot \left[\underset{\approx}{F}\,\underset{\sim}{n} + O\left(\|\Delta X\|\right)\right]$$

$$= (\underset{\approx}{F}\,\underset{\sim}{n}) \cdot (\underset{\approx}{F}\,\underset{\sim}{n}) + O\left(\|\Delta X\|\right)$$

$$\lambda^2 = \left(\underset{\approx}{F}^T \underset{\approx}{F}\,\underset{\sim}{n}\right) \cdot \underset{\sim}{n} + O\left(\|\Delta X\|\right)$$

$$\Rightarrow \lambda^2\left(\underset{\sim}{X}, \underset{\sim}{n}\right) = \lim_{\|\Delta X\| \to 0} \left(\underset{\approx}{F}^T \underset{\approx}{F}\,\underset{\sim}{n} \cdot \underset{\sim}{n} + O\left(\|\Delta X\|\right)\right)$$

$$= \left(\underset{\approx}{F}^T \underset{\approx}{F}\,\underset{\sim}{n}\right) \cdot \underset{\sim}{n}$$

$$\Rightarrow \epsilon\left(\underset{\sim}{X}, \underset{\sim}{n}\right) = \sqrt{\left(\underset{\approx}{F}^T \underset{\approx}{F}\,\underset{\sim}{n}\right) \cdot \underset{\sim}{n}} - 1$$

$$F_{ij} = \frac{\partial f_i}{\partial X_j} \rightarrow$$

$$\underset{\sim}{u}(\underset{\sim}{X}) = \underset{\sim}{x}(\underset{\sim}{X}) - \underset{\sim}{X} = \underset{\sim}{f}(\underset{\sim}{X}) - \underset{\sim}{X}$$

$$u_i = f_i - X_i$$

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial f_i}{\partial X_j} - \frac{\partial X_i}{\partial X_j} \Rightarrow \frac{\partial f_i}{\partial X_j} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

$$\Rightarrow \underset{\approx}{F} = \nabla \underset{\sim}{u} + \underset{\approx}{I}$$

$$\epsilon(\underset{\sim}{X}, \underset{\sim}{n}) = \sqrt{\left\{(\underline{\underline{I}} + \nabla \underline{u})^T (\underline{\underline{I}} + \nabla \underline{u})\, \underline{n}\right\} \cdot \underline{n}} - 1$$

$$= \sqrt{\left\{\left(\underline{\underline{I}} + \nabla \underline{u} + \nabla \underline{u}^T + \nabla \underline{u}^T \nabla \underline{u}\right) \underline{n}\right\} \cdot \underline{n}} - 1$$

$$= \sqrt{1 + \underbrace{\left\{(\nabla \underline{u} + \nabla \underline{u}^T)\underline{n}\right\} \cdot \underline{n}} + \cancel{\underbrace{\left\{(\nabla \underline{u}^T \nabla \underline{u})\underline{n}\right\} \cdot \underline{n}}}} - 1$$

$$\|\nabla \underline{u}\| << 1$$

$$\left\| \frac{\partial u_i}{\partial X_j} \right\| << 1$$

$$\approx \left(1 + \left\{(\nabla \underline{u} + \nabla \underline{u}^T)\underline{n}\right\} \cdot \underline{n}\right)^{1/2} - 1$$

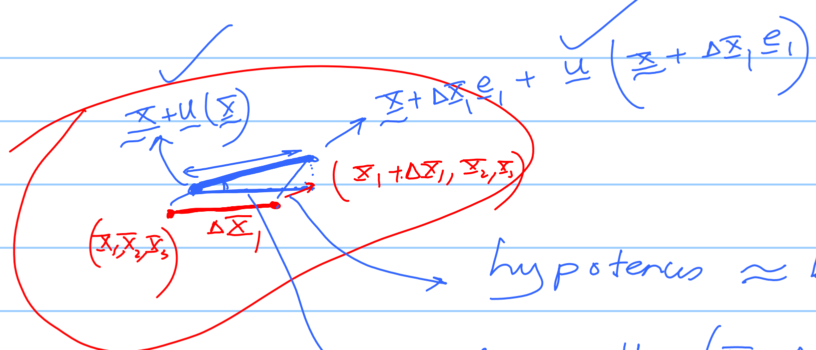
$$\approx \cancel{1} + \frac{1}{2}\left\{(\nabla \underline{u} + \nabla \underline{u}^T)\underline{n}\right\} \cdot \underline{n} + \cdots - \cancel{1}$$

$$\boxed{\epsilon(\underset{\sim}{X}, \underline{n}) = \frac{1}{2}\left\{(\nabla \underline{u} + \nabla \underline{u}^T)\underline{n}\right\} \cdot \underline{n}} \qquad (\|\nabla \underline{u}\| << 1)$$

$$\epsilon(\underset{\sim}{X}, \underline{n}) = \left(\underbrace{\begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right) \\ & \frac{\partial u_2}{\partial X_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right) \\ & & \frac{\partial u_3}{\partial X_3} \end{bmatrix}}_{\frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T)} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \right) \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

$$\epsilon(\underset{\sim}{X}, \underline{e}_1) = \left(\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{\partial u_1}{\partial X_1} \quad ✓✓$$



hypotenuse ≈ base

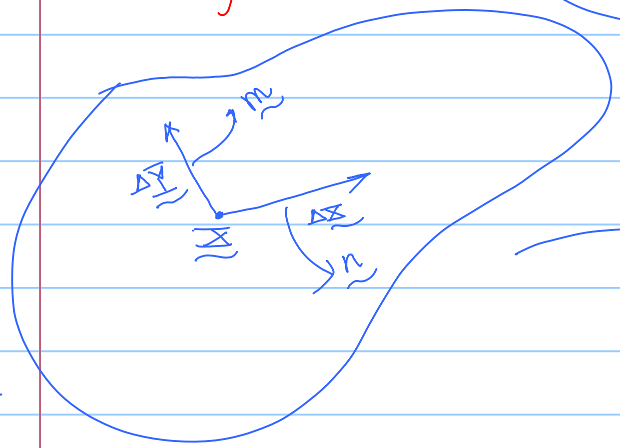
$$\Delta l = u_1(X_1 + \Delta X_1, X_2, X_3) - u_1(X_1, X_2, X_3)$$

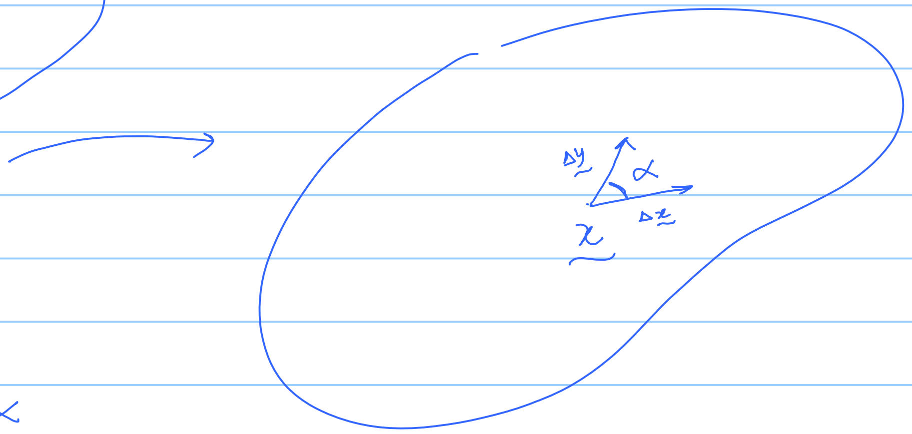
$$\approx \frac{\partial u_1}{\partial X_1} \Delta X_1$$

$$\epsilon = \frac{\Delta l}{L} = \frac{\partial u_1}{\partial X_1}\frac{\Delta X_1}{\Delta X_1} = \frac{\partial u_1}{\partial X_1}$$

Formula for shear strain

→ change in angle between lines ⊥ line elements!





$$\gamma = \pi/2 - \alpha$$

$$\Delta \underset{\sim}{x} = \underset{\approx}{F}\,\Delta \underset{\sim}{X} + O\left(\|\Delta \underset{\sim}{X}\|^2\right)$$

$$\Delta \underset{\sim}{y} = \underset{\approx}{F}\,\Delta \underset{\sim}{Y} + O\left(\|\Delta \underset{\sim}{Y}\|^2\right)$$

$$\cos\alpha = \frac{\Delta \underset{\sim}{x}\cdot\Delta \underset{\sim}{y}}{\|\Delta \underset{\sim}{x}\|\;\|\Delta \underset{\sim}{y}\|}$$

$$= \frac{\left\{\underset{\approx}{F}\,\Delta \underset{\sim}{X} + O\left(\|\Delta \underset{\sim}{X}\|^2\right)\right\}\cdot\left\{\underset{\approx}{F}\,\Delta \underset{\sim}{Y} + O\left(\|\Delta \underset{\sim}{Y}\|^2\right)\right\}}{\lambda(\underset{\sim}{X},\underset{\sim}{n})\,\|\Delta \underset{\sim}{X}\| \quad \lambda(\underset{\sim}{X},\underset{\sim}{m})\,\|\Delta \underset{\sim}{Y}\|}$$

$$= \frac{\left\{\underset{\approx}{F}\underset{\sim}{n} + O\left(\|\Delta \underset{\sim}{X}\|\right)\right\}\cdot\left\{\underset{\approx}{F}\underset{\sim}{m} + O\left(\|\Delta \underset{\sim}{Y}\|\right)\right\}}{\lambda(\underset{\sim}{X},\underset{\sim}{n})\quad \lambda(\underset{\sim}{X},\underset{\sim}{m})}$$

$$\operatorname{lt}_{\|\Delta \underset{\sim}{X}\|\to 0,\ \|\Delta \underset{\sim}{Y}\|\to 0}$$

$$\cos\alpha = \frac{\underset{\approx}{F}\underset{\sim}{n}\cdot\underset{\approx}{F}\underset{\sim}{m}}{\lambda(\underset{\sim}{X},\underset{\sim}{n})\,\lambda(\underset{\sim}{X},\underset{\sim}{m})} = \frac{\left\{\left(\underset{\approx}{I} + \nabla\underset{\sim}{u} + \nabla\underset{\sim}{u}^T + \nabla\underset{\sim}{u}^T\nabla\underset{\sim}{u}\right)\underset{\sim}{n}\right\}\cdot\underset{\sim}{m}}{\lambda(\underset{\sim}{X},\underset{\sim}{n})\quad \lambda(\underset{\sim}{X},\underset{\sim}{m})}$$

$$\|\nabla\underset{\sim}{u}\| << 1$$

$$\cos\alpha \approx \left\{\left(\nabla\underset{\sim}{u} + \nabla\underset{\sim}{u}^T\right)\underset{\sim}{n}\right\}\cdot\underset{\sim}{m}\quad \left(1+\epsilon(\underset{\sim}{X},\underset{\sim}{n})\right)^{-1}\left(1+\epsilon(\underset{\sim}{X},\underset{\sim}{m})\right)^{-1}$$

$$= \left\{ \left(\underline{\nabla u} + \underline{\nabla u} \right)^T \underset{\sim}{n} \right\} \cdot \underset{\sim}{m} \; \left(1 - \epsilon (\underset{\sim}{\underline{x}}, \underset{\sim}{n}) + \cdots \right) \left(1 - \epsilon (\underset{\sim}{\underline{x}}, \underset{\sim}{m}) \cdots \right)$$

$$\cos \alpha \simeq \left\{ \left(\underline{\nabla u} + \underline{\nabla u}^T \right) \underset{\sim}{n} \right\} \cdot \underset{\sim}{m}$$