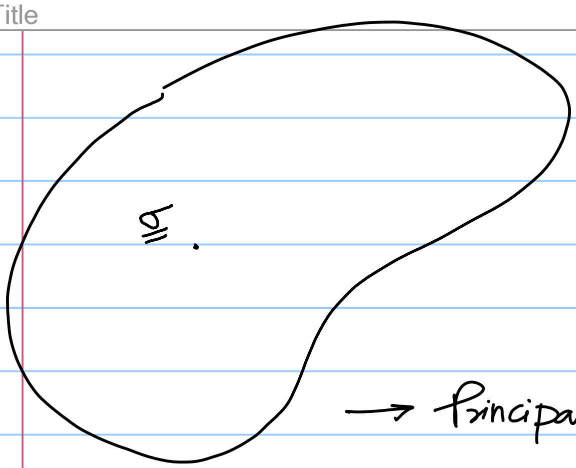
Note Title 9/2/2022

Lecture 13 Stress invariants



→ Stress tensor is invariant w.r.t. coordinate system

→ Stress matrix is not an invariant!

→ Principal stress components are invariant!

Suppose in $(\underline{e}_1 - \underline{e}_2 - \underline{e}_3)$

$$[\underline{\underline{\sigma}}][\underline{n}] = \lambda [\underline{n}]$$

$$\Rightarrow [\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}][\underline{n}] = [\underline{0}]$$

$$\Rightarrow \begin{bmatrix} \sigma_{11}-\lambda & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22}-\lambda & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33}-\lambda \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\det = 0 \Rightarrow \det\left([\underline{\underline{\sigma}}] - \lambda [\underline{\underline{I}}]\right) = 0$$

$\Downarrow$

Characteristic equation!

In $(\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3)$ coordinate system

$$\boxed{[\hat{\underline{\underline{\sigma}}}] = [\underline{\underline{R}}]^T [\underline{\underline{\sigma}}][\underline{\underline{R}}]}$$

$$\det\left[[\hat{\underline{\underline{\sigma}}}] - \hat{\lambda}[\underline{\underline{I}}]\right] = 0$$

$$\Rightarrow \det\left[[\underline{\underline{R}}]^T [\underline{\underline{\sigma}}][\underline{\underline{R}}] - \hat{\lambda} \underbrace{[\underline{\underline{R}}]^T[\underline{\underline{R}}]}_{[\underline{\underline{I}}]}\right] = 0$$

$$\Rightarrow \det\left[[\underline{\underline{R}}]^T \left([\underline{\underline{\sigma}}] - \hat{\lambda}[\underline{\underline{I}}]\right)[\underline{\underline{R}}]\right] = 0$$

$$\Rightarrow \det[\underline{\underline{R}}]^T \det\left([\underline{\underline{\sigma}}] - \hat{\lambda}[\underline{\underline{I}}]\right) \det[\underline{\underline{R}}] = 0$$

$$\Rightarrow \det\left([\underline{\underline{\sigma}}] - \hat{\lambda}[\underline{\underline{I}}]\right) = 0 \quad ✓ \qquad \Rightarrow \lambda = \hat{\lambda}$$

for a matrix $[\underline{\underline{A}}]$ & $[\underline{\underline{B}}]$

$\lambda([\underline{\underline{A}}]) = \lambda\left([\underline{\underline{B}}][\underline{\underline{A}}][\underline{\underline{B}}]^{-1}\right)$ ✓

similarity transfr.

$$\det\left([\underline{\underline{\sigma}}] - \lambda[\underline{\underline{I}}]\right) = 0$$

$$\Rightarrow -\lambda^3 + \lambda^2(\sigma_{11} + \sigma_{22} + \sigma_{33}) - \lambda(\sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} + \sigma_{11}\sigma_{22} - \tau_{12}^2 - \tau_{13}^2 - \tau_{23}^2) + \det([\underline{\underline{\sigma}}]) = 0$$

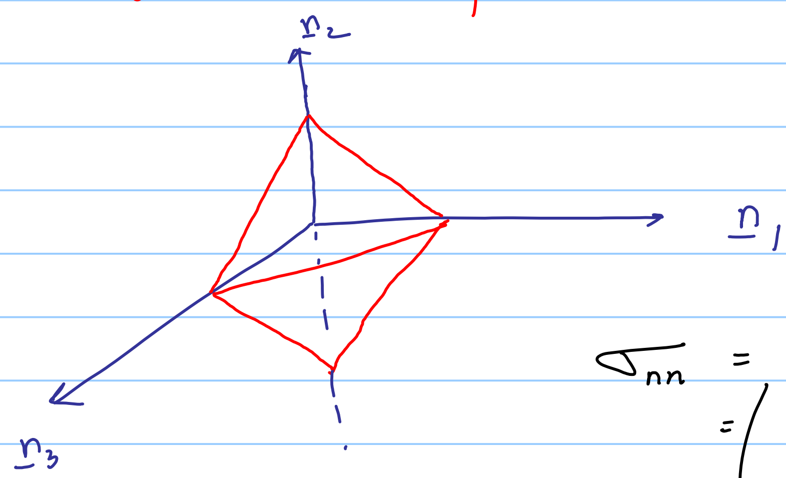
$$\det\left([\hat{\underline{\underline{\sigma}}}] - \lambda[\underline{\underline{I}}]\right) = 0$$

$$\Rightarrow -\lambda^3 + \lambda^2\underbrace{(\hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33})}_{I_1} - \lambda\overbrace{(\hat{\sigma}_{22}\hat{\sigma}_{33} + \hat{\sigma}_{11}\hat{\sigma}_{33} + \hat{\sigma}_{11}\hat{\sigma}_{22} - \hat{\tau}_{12}^2 - \hat{\tau}_{13}^2 - \hat{\tau}_{23}^2)}^{I_2} + \underbrace{\det([\hat{\underline{\underline{\sigma}}}])}_{I_3} = 0$$

$$\Rightarrow \sigma_{11} + \sigma_{22} + \sigma_{33} = \hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33}$$

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \Rightarrow \begin{aligned} I_1 &= \lambda_1 + \lambda_2 + \lambda_3 \\ I_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\ I_3 &= \lambda_1\lambda_2\lambda_3 \end{aligned}$$

Octahedral stress component



$$\underline{n} = \begin{bmatrix} \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\sigma_{nn} = (\underline{\underline{\sigma}}\,\underline{n}) \cdot \underline{n}$$

$$= \left(\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \end{bmatrix}\right) \cdot \begin{bmatrix} \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\sigma_{oct} = \frac{\lambda_1 + \lambda_2 + \lambda_3}{3} = I_1/3$$

$$\tau_{oct}^2 = \|\underline{\underline{\sigma}}\,\underline{n}\|^2 - \sigma_{oct}^2$$

$$= \left\| \begin{bmatrix} \pm\frac{1}{\sqrt{3}}\lambda_1 \\ \pm\frac{1}{\sqrt{3}}\lambda_2 \\ \pm\frac{1}{\sqrt{3}}\lambda_3 \end{bmatrix} \right\|^2 - \left(\frac{I_1}{3}\right)^2$$

$$= \frac{1}{3}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - \frac{1}{9}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 + 2\lambda_1\lambda_3 + 2\lambda_2\lambda_3)$$

$$= \frac{3\lambda_1^2 + 3\lambda_2^2 + 3\lambda_3^2 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 - 2\lambda_1\lambda_2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3}{9}$$

$$\tau_{oct}^2 = \frac{2(\lambda_1+\lambda_2+\lambda_3)^2 - 6(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)}{9}$$

$$= \frac{2}{9}\left[I_1^2 - 3I_2\right] \quad \text{or,} \quad \boxed{|\tau_{oct}| = \frac{\sqrt{2}}{3}\sqrt{I_1^2 - 3I_2}}$$

Decomposition of stress tensor

$$\underline{\underline{\sigma}} = \underbrace{\frac{1}{3} I_1(\underline{\underline{\sigma}})\, \underline{\underline{I}}}_{\text{hydrostatic part of stress tensor}} + \underbrace{\left[\underline{\underline{\sigma}} - \frac{1}{3} I_1(\underline{\underline{\sigma}})\, \underline{\underline{I}}\right]}_{\underline{\underline{\sigma}}^d}$$

state of pure shear!!

deviatoric part of stress tensor

$$p_{eq} = -\frac{1}{3} I_1(\underline{\underline{\sigma}})$$

$$I_1(\underline{\underline{\sigma}}^d) = I_1(\underline{\underline{\sigma}}) - \frac{1}{3} I_1(\underline{\underline{\sigma}})\, I_1(\underline{\underline{I}})$$

$$= I_1(\underline{\underline{\sigma}}) - \frac{1}{3} I_1(\underline{\underline{\sigma}})\, 3$$

$$= 0 \,!$$

$$\begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{bmatrix} = \frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{2\sigma_{11}}{3} - \left(\frac{\sigma_{22}+\sigma_{33}}{3}\right) & \tau_{12} & \tau_{13} \\ \tau_{12} & \frac{2\sigma_{22}}{3} - \left(\frac{\sigma_{11}+\sigma_{33}}{3}\right) & \tau_{23} \\ \tau_{13} & \tau_{23} & X \end{bmatrix}$$