# Lecture 11 (Maximizing shear comp. of traction)

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

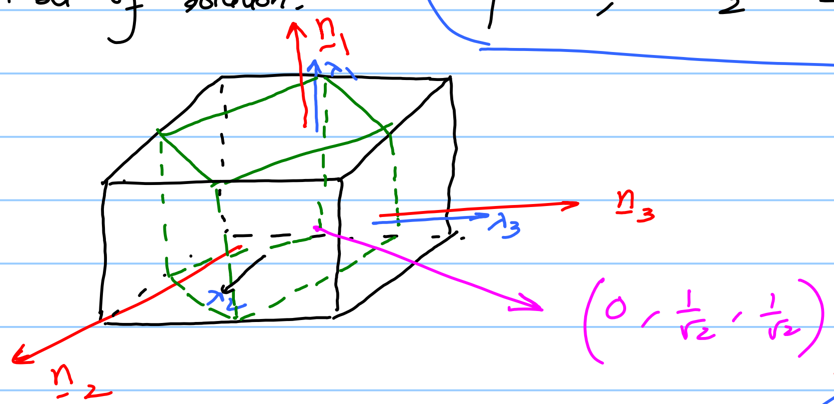
$$\sigma_{nn} = (\underline{\underline{\sigma}}\,\underline{n}) \cdot \underline{n} = \lambda_1 n_1^2 + \lambda_2 n_2^2 + \lambda_3 n_3^2$$

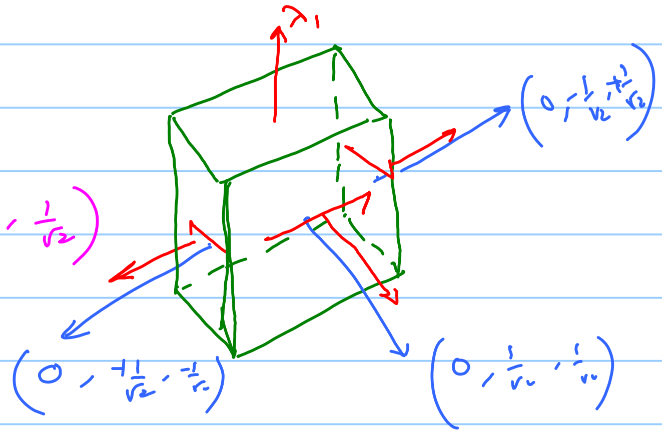
$$\tau^2 = \|\underline{\underline{\sigma}}\,\underline{n}\|^2 - \sigma_{nn}^2 = \lambda_1^2 n_1^2 + \lambda_2^2 n_2^2 + \lambda_3^2 n_3^2 - (\lambda_1 n_1^2 + \lambda_2 n_2^2 + \lambda_3 n_3^2)^2$$

$$\Rightarrow \sigma_{nn} = \frac{\lambda_2 + \lambda_3}{2}$$

$$\Rightarrow \tau = \left|\frac{\lambda_2 - \lambda_3}{2}\right|$$

A set of solution: $n_1 = 0$, $n_2 = \pm\frac{1}{\sqrt{2}}$, $n_3 = \pm\frac{1}{\sqrt{2}}$




$\underline{n} = (0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$, $\sigma_{nn} = \frac{\lambda_2 + \lambda_3}{2}$, $\tau = \left|\frac{\lambda_2 - \lambda_3}{2}\right|$

$\underline{n} = (\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}})$, $\sigma_{nn} = \frac{\lambda_1 + \lambda_3}{2}$, $\tau = \left|\frac{\lambda_1 - \lambda_3}{2}\right|$

$\underline{n} = (\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0)$, $\sigma_{nn} = \frac{\lambda_1 + \lambda_2}{2}$, $\tau = \left|\frac{\lambda_1 - \lambda_2}{2}\right|$

→ 12 such planes!

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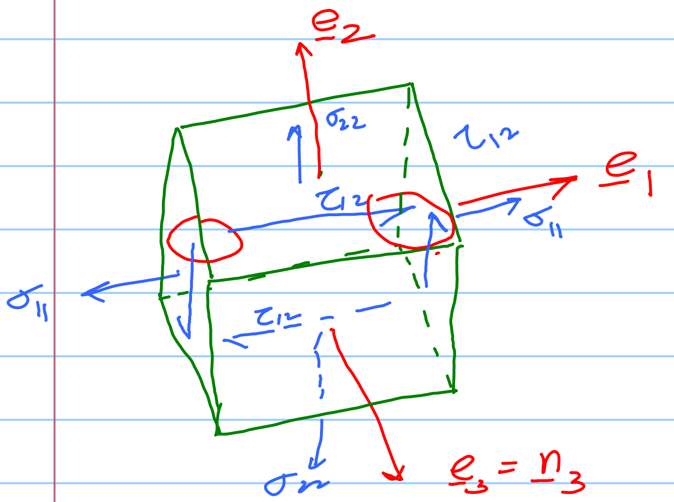
## Mohr's Circle

→ A geometrical/graphical approach to obtain principal stress components, principal normals, max. shear and their planes!

→ Scope is limited :- we are searching only those planes whose normal is $\perp$ to at least one of the principal axes!

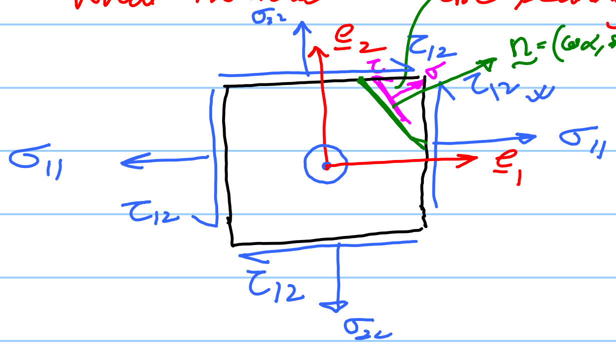
* Let us suppose that principal axis is $\underline{n}_3$! ← third principal axis!

* Let's choose $(\underline{e}_1 - \underline{e}_2 - \underline{e}_3)$ coordinate system such that $\underline{e}_3 = \underline{n}_3$



$$[\underline{\underline{\sigma}}]_{(\underline{e}_1, \underline{e}_2, \underline{e}_3 = \underline{n}_3)} = \begin{bmatrix} \sigma_{11} & \tau_{12} & 0 \\ \tau_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

What normals we are scanning:



$\underline{n}^{\perp} = (-\sin\alpha, \cos\alpha, 0)$

$\underline{n} = (\cos\alpha, \sin\alpha, 0)$

$$[\underline{n}]_{(\underline{e}_1 - \underline{e}_2 - \underline{e}_3 = \underline{n}_3)} = \begin{bmatrix} n_1 \\ n_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix}$$

rotate $\underline{e}_1$-plane by $\alpha$ about $\underline{e}_3$ axis!

$$\sigma = (\underline{\underline{\sigma}}\,\underline{n}) \cdot \underline{n} = \left( \underbrace{\begin{bmatrix} \sigma_{11} & \tau_{12} & 0 \\ \tau_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ 0 \end{bmatrix}} \right) \cdot \begin{bmatrix} n_1 \\ n_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11}\cos\alpha + \tau_{12}\sin\alpha \\ \tau_{12}\cos\alpha + \sigma_{22}\sin\alpha \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix}$$

$$= \sigma_{11}\cos^2\alpha + 2\tau_{12}\sin\alpha\cos\alpha + \sigma_{22}\sin^2\alpha \quad \checkmark$$

$$\tau = (\underline{\underline{\sigma}}\,\underline{n}) \cdot \underline{n}^{\perp} = \begin{bmatrix} \sigma_{11}\cos\alpha + \tau_{12}\sin\alpha \\ \tau_{12}\cos\alpha + \sigma_{22}\sin\alpha \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix}$$

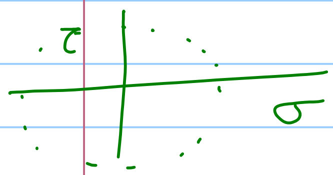
$$= -\sigma_{11}\sin\alpha\cos\alpha - \tau_{12}\sin^2\alpha + \tau_{12}\cos^2\alpha + \sigma_{22}\sin\alpha\cos\alpha$$

$$= -(\sigma_{11} - \sigma_{22})\sin\alpha\cos\alpha + \tau_{12}(\cos^2\alpha - \sin^2\alpha)$$

$$\sigma = \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}}{2}\left(2\cos^2\alpha - 1\right) - \frac{\sigma_{22}}{2}\left(1 - 2\sin^2\alpha\right) + 2\tau_{12}\sin\alpha\cos\alpha$$

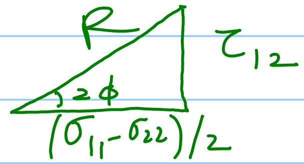
$$= \frac{\sigma_{11}+\sigma_{22}}{2} + \cos 2\alpha\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right) + \tau_{12}\sin 2\alpha$$

$$\tau = -\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)\sin 2\alpha + \tau_{12}\cos 2\alpha$$




$$R = \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \tau_{12}^2}$$

$$\tan 2\phi = \frac{\tau_{12}}{\frac{\sigma_{11}-\sigma_{22}}{2}}$$




$$\sigma = \frac{\sigma_{11}+\sigma_{22}}{2} + R\left[\cos 2\alpha\cos 2\phi + \sin 2\alpha\sin 2\phi\right]$$

$$\tau = R\left[-\sin 2\alpha\cos 2\phi + \cos 2\alpha\sin 2\phi\right]$$

$$\sigma = \frac{\sigma_{11}+\sigma_{22}}{2} + R\cos\left(2\phi - 2\alpha\right)$$

$$\tau = R\sin\left(2\phi - 2\alpha\right)$$