Lecture 10

Note Title 8/26/2022

Principal planes and principal stress components

$$\sigma_{nn} = \underset{\sim}{t}^n \cdot \underset{\sim}{n} = (\underset{\approx}{\sigma}\, \underset{\sim}{n}) \cdot \underset{\sim}{n} = \sum_i \sum_j \sigma_{ij}\, n_j\, n_i .$$

$$f = \sum_i \sum_j \sigma_{ij}\, n_i n_j - \lambda \left(\sum_i n_i n_i - 1 \right)$$

$$\underset{\approx}{\sigma}\, \underset{\sim}{n} = \lambda\, \underset{\sim}{n}$$

↑ principal planes → principal stress component

→ no shear component of traction on principal plane

→ The fact that $\underset{\approx}{\sigma}$ is symmetric implies that all its eigenvalues are real!

$$\underbrace{\underset{\approx}{\sigma}\, \underset{\sim}{n}_1 = \lambda_1 \underset{\sim}{n}_1}_{\text{dot with } \underset{\sim}{n}_2} , \quad \underbrace{\underset{\approx}{\sigma}\, \underset{\sim}{n}_2 = \lambda_2 \underset{\sim}{n}_2}_{\text{dot with } \underset{\sim}{n}_1}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$(\underset{\approx}{\sigma}\, \underset{\sim}{n}_1) \cdot \underset{\sim}{n}_2 = \lambda_1\, \underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2 \qquad (\underset{\approx}{\sigma}\, \underset{\sim}{n}_2) \cdot \underset{\sim}{n}_1 = \lambda_2 (\underset{\sim}{n}_2 \cdot \underset{\sim}{n}_1)$$

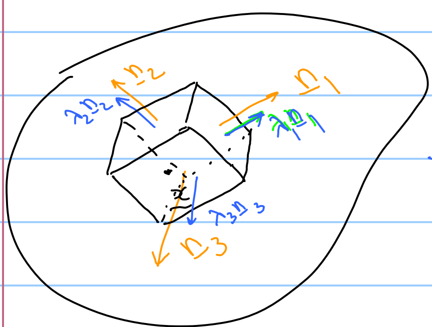
$$\Downarrow$$

$$(\underset{\sim}{n}_1 \cdot \underset{\approx}{\sigma}\, \underset{\sim}{n}_2) = \lambda_1\, \underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2$$

subtract

$$0 = (\lambda_1 - \lambda_2)(\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2)$$

* If $\lambda_1, \lambda_2, \lambda_3$ are all different, then $(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$ form an orthonormal triad!



$$\Rightarrow \quad [\underset{\approx}{\sigma}]_{(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

What if $(\lambda_1 = \lambda_2)$ are the same?
$\qquad\qquad \overset{\shortparallel}{\lambda}$

$$\alpha_1 \left(\underset{\approx}{\sigma}\, \underset{\sim}{n}_1 = \lambda\, \underset{\sim}{n}_1 \right), \quad \left(\underset{\approx}{\sigma}\, \underset{\sim}{n}_2 = \lambda\, \underset{\sim}{n}_2 \right) \alpha_2$$

sum

$$\alpha_1\, \underset{\approx}{\sigma}\, \underset{\sim}{n}_1 = \alpha_1 \lambda\, \underset{\sim}{n}_1$$
$$+$$
$$\alpha_2\, \underset{\approx}{\sigma}\, \underset{\sim}{n}_2 = \alpha_2 \lambda\, \underset{\sim}{n}_2$$

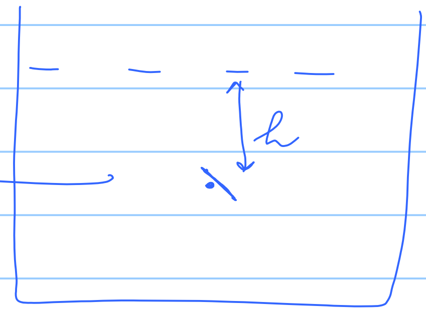
$$\Longrightarrow \underset{\approx}{\sigma} \overbrace{\left(\alpha_1 \underset{\sim}{n}_1 + \alpha_2 \underset{\sim}{n}_2 \right)}^{\text{eigenvector}} = \lambda \left(\alpha_1 \underset{\sim}{n}_1 + \alpha_2 \underset{\sim}{n}_2 \right)$$

all planes with normal $\underset{\shortparallel}{\alpha_1} \underline{n}_1 + \underset{\shortparallel}{\alpha_2} \underline{n}_2$ are also principal planes

* If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, all planes are principal planes!

$$\underline{\underline{\sigma}}\,\underline{n} = \lambda\,\underline{n}$$

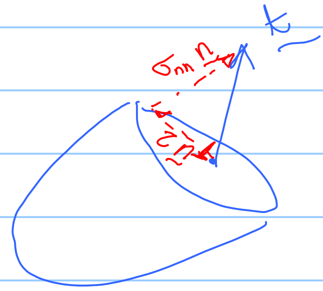
$$\underline{t}^n = \underbrace{(-\rho g h)}_{\lambda}\,\underline{n}$$



$$\boxed{\underline{\underline{\sigma}} = -\rho g h\,\underline{\underline{I}}}$$

Maximising the shear component of traction

$$\underline{t} = \underline{\underline{\sigma}}\,\underline{n}, \quad \sigma_{nn} = (\underline{\underline{\sigma}}\,\underline{n})\cdot\underline{n}$$



$$\underline{t} = \sigma_{nn}\,\underline{n} + \underbrace{(\underline{t} - \sigma_{nn}\,\underline{n})}_{\tau\,\underline{n}^{\perp}}$$

$$\tau^2 = \|\underline{t}\|^2 - \sigma_{nn}^2 \quad \text{(Using Pythagoras theorem)}$$

$$= \|\underline{\underline{\sigma}}\,\underline{n}\|^2 - (\underline{\underline{\sigma}}\,\underline{n}\cdot\underline{n})^2$$

let us choose the coordinate system to be the ones corresponding to principal normals!

$$\tau^2 = \left\| \begin{bmatrix} \lambda_1 n_1 \\ \lambda_2 n_2 \\ \lambda_3 n_3 \end{bmatrix} \right\|^2 - (\lambda_1 n_1^2 + \lambda_2 n_2^2 + \lambda_3 n_3^2)^2$$

$$= \lambda_1^2 n_1^2 + \lambda_2^2 n_2^2 + \lambda_3^2 n_3^2$$

$$\tau^2 = \sum_i \lambda_i^2 n_i^2 - \left(\sum_i \lambda_i n_i^2\right)^2$$

$$f = \sum_i \lambda_i^2 n_i^2 - \left(\sum_i \lambda_i n_i^2\right)^2 + \alpha\left(\sum_i n_i^2 - 1\right)$$

$$\frac{\partial f}{\partial n_k} = \sum_i \lambda_i^2 \, 2 n_i \delta_{ik} - 2\left(\sum_i \lambda_i n_i^2\right)\left(\sum_i 2\lambda_i n_i \delta_{ik}\right) + \alpha \sum_i 2 n_i \delta_{ik} = 0$$

$$= 2\lambda_k^2 n_k - 4\lambda_k n_k \left(\sum_i \lambda_i n_i^2\right) + 2\alpha n_k = 0$$

$$\Rightarrow n_k \left(\lambda_k^2 - 2\lambda_k \sum_i \lambda_i n_i^2 + \alpha \right) = 0$$

$$n_1 \left(\lambda_1^2 - 2\lambda_1 \left(\underline{\lambda_1 n_1^2} + \lambda_2 n_2^2 + \lambda_3 n_3^2\right) + \alpha\right) = 0$$

$$n_2 \left(- \quad - \quad - \quad - \right) = 0$$

$$n_3 \left(- \quad - \quad - \quad - \right) = 0$$

$$\Rightarrow n_1 = 0$$

$$\Rightarrow \lambda_2^2 - 2\lambda_2\left(\lambda_2 n_2^2 + \lambda_3 n_3^2\right) + \alpha = 0$$

$$\Rightarrow \lambda_3^2 - 2\lambda_3\left(\lambda_2 n_2^2 + \lambda_3 n_3^2\right) + \alpha = 0$$

$$\left(\lambda_2^2 - \lambda_3^2\right) - 2\left(\lambda_2 - \lambda_3\right)\left(\lambda_2 n_2^2 + \lambda_3 n_3^2\right) = 0$$

$$\left(\lambda_2 - \lambda_3\right)\left[\left(\lambda_2 + \lambda_3\right) - 2\left(\lambda_2 n_2^2 + \lambda_3 n_3^2\right)\right] = 0$$

$$= \lambda_2\left(1 - 2n_2^2\right) + \lambda_3\left(1 - 2n_3^2\right) = 0$$

$\Downarrow$ arbitrary (λ_2, λ_3)

$$\boxed{n_1 = 0 \qquad n_2 = \pm\frac{1}{\sqrt{2}}, \quad n_3 = \pm\frac{1}{\sqrt{2}}}$$