Tutorial 9

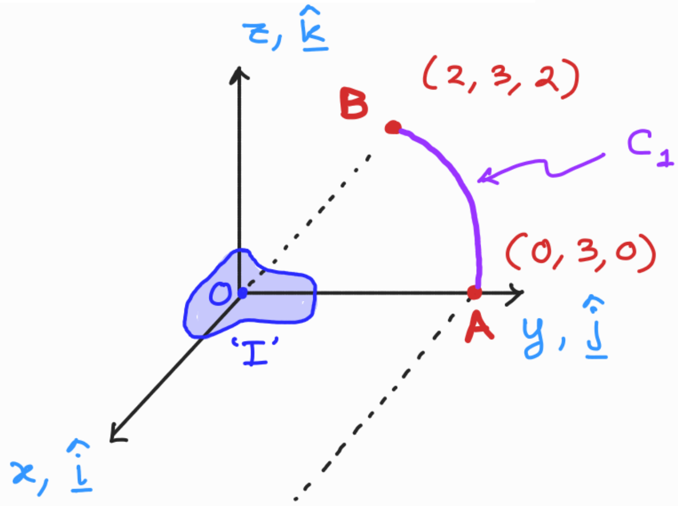
(Part A)

1) Is the following force conservative?

$$\boldsymbol{F} = (-2xy + yz)\hat{\boldsymbol{i}} + (-x^2 + xz - z)\hat{\boldsymbol{j}} + (xy - y)\hat{\boldsymbol{k}}$$

(a) If it is conservative, find its potential function V

(b) Find the work done by this force in moving a particle (say P) along an open quarter circular path C_1 (start at A and end at B)



Soln :

(a) To check if the force is conservative or not, we use the check the curl of the force:

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2xy + yz & -x^2 + xz - z & xy - y \end{vmatrix}$$

$$\underline{\nabla} = \left(\frac{\partial}{\partial x}\hat{\underline{i}} + \frac{\partial}{\partial y}\hat{\underline{j}} + \frac{\partial}{\partial z}\hat{\underline{k}} \right)$$

$$\underline{\nabla} \times \underline{F} = (\cancel{x}-\cancel{1}-\cancel{x}+\cancel{1})\,\hat{\underline{i}} - (\cancel{y}-\cancel{y})\,\hat{\underline{j}} + (-\cancel{2x}+\cancel{z}+\cancel{2x}-\cancel{z})\,\hat{\underline{k}}$$

$$= \underline{0} \quad \forall \; x, y, z$$

for all

$\Rightarrow$ $\underline{F}$ is CONSERVATIVE !

$\Rightarrow$ Work done by this force $\underline{F}$ in moving the particle P from pt A to pt B is independent of the path C_1.

(b) The potential function $V(x,y,z)$ is related to the conservative force $\underline{F}$ as:

$$\underbrace{F_x = -\frac{\partial V}{\partial x}}_{(i)}, \quad \underbrace{F_y = -\frac{\partial V}{\partial y}}_{(ii)}, \quad \underbrace{F_z = -\frac{\partial V}{\partial z}}_{(iii)}$$

Integrate (i) :

$$-2xy + yz = -\frac{\partial V}{\partial x}$$

$$\Rightarrow \int (-2xy + yz)\, dx = -V(x,y,z) + \underbrace{f(y,z)}_{\text{const w.r.t } x}$$

$$\Rightarrow \quad V(x,y,z) = x^2 y - xyz + f(y,z) \quad \text{---} \quad (i)$$

Let's now use this expression of $V(x,y,z)$ in (ii) and (iii)

$$-x^2 + xz - z = -\frac{\partial V}{\partial y} = -\frac{\partial}{\partial y}\left(x^2 y - xyz + f(y,z)\right)$$

$$\Rightarrow \quad \cancel{-x^2} + \cancel{xz} - z = -\cancel{x^2} + \cancel{xz} - \frac{\partial f}{\partial y}$$

$$\Rightarrow \quad \frac{\partial f}{\partial y} = z \quad \Rightarrow \quad f(y,z) = yz + \underbrace{c(z)}_{\text{const. w.r.t. } y}$$

To determine the value of the constant c, use (iii)

$$F_z = -\frac{\partial V}{\partial z}$$

$$\Rightarrow \quad xy - y = -\frac{\partial}{\partial z}\left(x^2y - xyz + yz + c(z)\right)$$

$$\Rightarrow \quad \cancel{xy} - \cancel{y} = \cancel{xy} - \cancel{y} + \frac{\partial c}{\partial z}$$

$$\Rightarrow \quad \frac{\partial c}{\partial z} = 0 \quad \Rightarrow \quad c = \text{constant}$$

$$\therefore \quad V(x,y,z) = x^2y - xyz + yz + c$$

(c) Work done

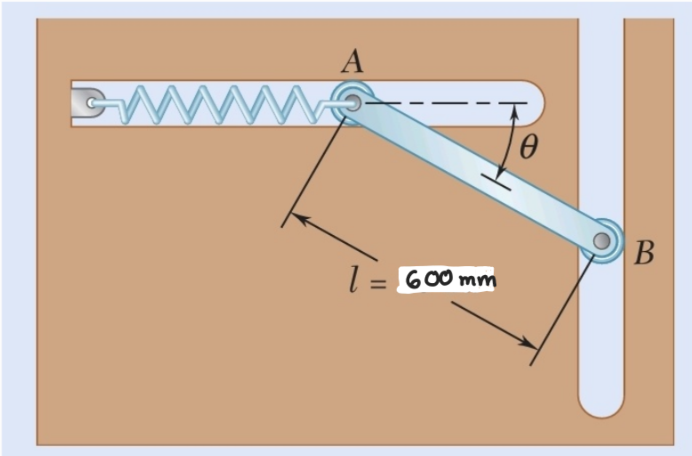
$$W_{A\to B} = -\left[V(\underline{r}_B) - V(\underline{r}_A)\right]$$

$$= -\left[V(2,3,2) - V(0,3,0)\right]$$

$$= -\left[\left(2^2\cdot 3 - 2\cdot 3\cdot 2 + 3\cdot 2 + c\right) - \left(0^2\cdot 3 - 0\cdot 3\cdot 0 + 3\cdot 0 + c\right)\right] = -6 \text{ Nm}$$

2) **17.39** The ends of a 4.5 kg rod AB are constrained to move along slots cut in a vertical plate as shown. A spring of constant k = 600 N/m is attached to end A in such a way that its tension is zero when $\theta = 0$. If the rod is released from rest when θ = 0°, determine the angular velocity of the rod and the velocity of end B when θ = 30°

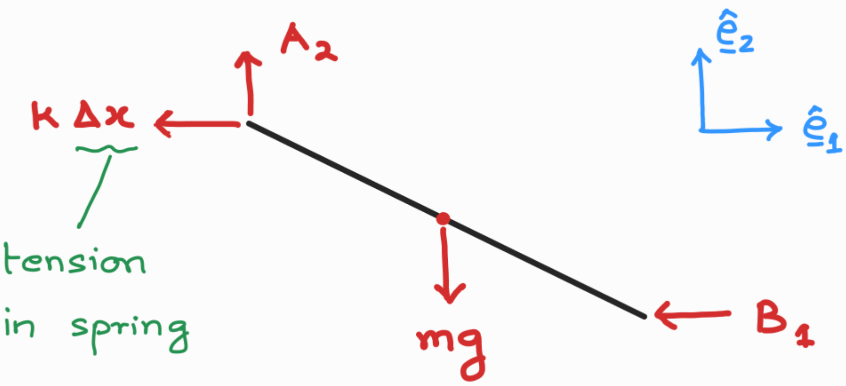
Neglect friction !



Soln : Let mass of rod be 'm'

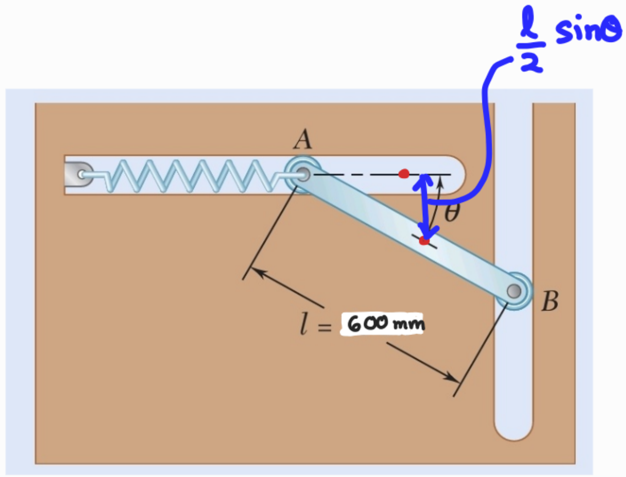
spring constant be 'k'

Let's draw the FBD of the rod AB



- A_2 and B_1 are workless forces
- Only the spring force and the gravitational weight mg do work. Both these forces are conservative forces.

$\Rightarrow$ Can make use of conservation of total energy



$$\Delta(T+V) = 0$$

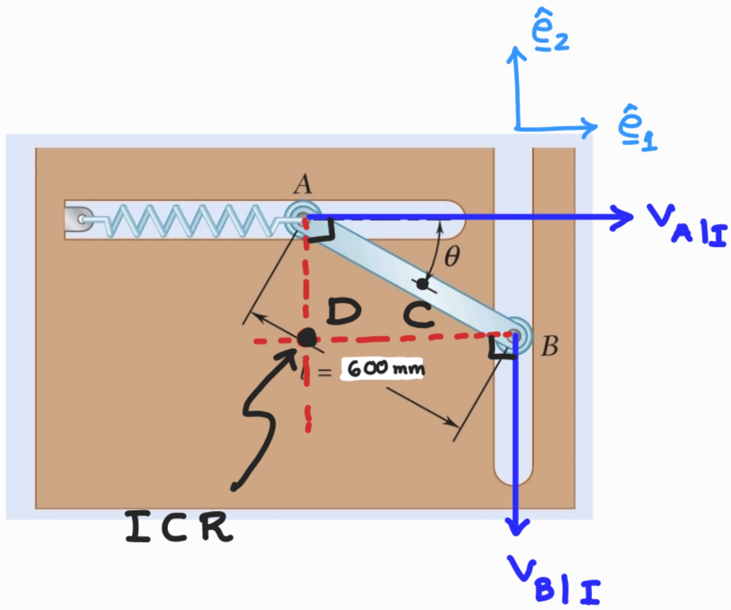
$$\Rightarrow \quad \Delta V + \Delta T = 0$$

Change in potential energy, ΔV

$$= V^{\text{spring force}} + V^{\text{gravitational force}}$$

$$= \left(\frac{1}{2}k\left[l(1-\cos\theta)\right]^2\right) + \left(-mg\,\frac{l}{2}\sin\theta\right)$$

Change in kinetic energy



The motion starts from rest

$\therefore \quad T_1 = 0$

Point D is the instantaneous center of rotation of the rod

The velocity of the COM C of the rod is:

$$\underline{v}_{C|I} = \underbrace{\underline{v}_{D|I}}_{0} + \underline{\omega}_{AB} \times \underline{r}_{CD}$$

$$= \omega\,\hat{e}_3 \times \left(\frac{l}{2}\cos\theta\,\hat{e}_1 + \frac{l}{2}\sin\theta\,\hat{e}_2\right)$$

$$= \frac{\omega l}{2}\left(-\sin\theta\ \hat{e}_1 + \cos\theta\ \hat{e}_2\right)$$

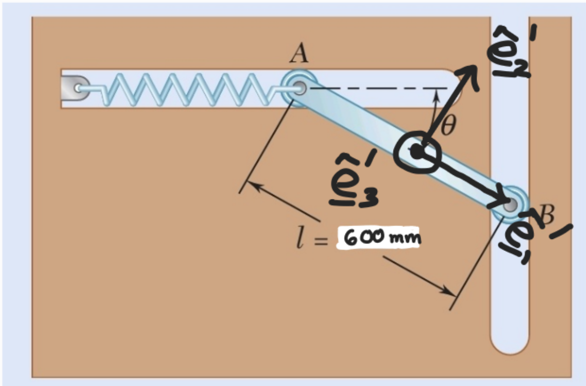
$$\therefore \quad \underline{v}_{C|I} \cdot \underline{v}_{C|I} = |\underline{v}_{C|I}|^2 = \left(\frac{\omega l}{2}\right)^2 = \frac{\omega^2 l^2}{4}$$

$$T_2 = \frac{1}{2} m\, \underline{v}_{C|I} \cdot \underline{v}_{C|I} + \underbrace{\frac{1}{2}\, \underline{\omega}_{AB} \cdot \underline{\underline{I}}^C\, \underline{\omega}_{AB}}_{\text{have to use a csys}}$$

$$= \frac{1}{2} m \frac{\omega^2 l^2}{4} +$$

$$\frac{1}{2} \omega^2 \frac{m l^2}{12}$$

$$= \frac{m \omega^2 l^2}{6}$$



$$[\underline{\underline{I}}^C]_{\begin{bmatrix}\hat{e}_1' \\ \hat{e}_2' \\ \hat{e}_3'\end{bmatrix}} = \begin{bmatrix} \frac{ml^2}{12} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{ml^2}{12} \end{bmatrix}$$

$$[\underline{\omega}_{AB}]_{\begin{bmatrix}\hat{e}_1' \\ \hat{e}_2' \\ \hat{e}_3'\end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$

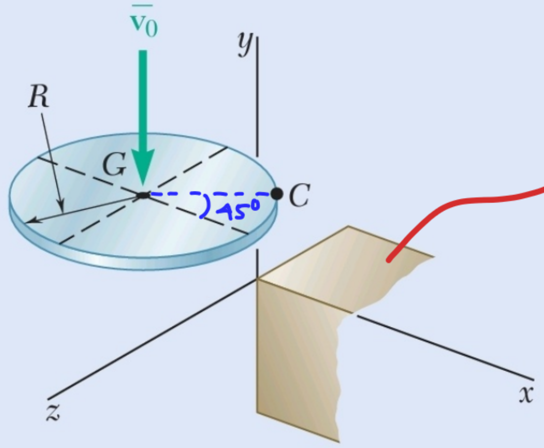
Now use $\Delta V + \Delta T = 0$

$$\Rightarrow \quad \frac{m l^2 \omega^2}{6} = m g \frac{l}{2} \sin\theta - \frac{k l^2}{2}(1-\cos\theta)^2$$

$$\Rightarrow \quad \omega = \sqrt{\frac{6}{m l^2}\left[\frac{m g l \sin\theta}{2} - \frac{k l^2}{2}(1-\cos\theta)^2\right]}$$

3) **18.29** A circular plate of mass m is falling with a velocity $\overline{\mathbf{v}}_0$ and no angular velocity when its edge C strikes an obstruction. A line passing the origin and parallel to the line CG makes a 45° angle with the x-axis. Assuming the impact to be perfectly plastic ($e = 0$), determine the angular velocity of the plate immediately after the impact.

RB ①



massive RB ② (unaffected by impulse)

Soln: Unconstrained collision and RB ② is massive

↳ can avoid calculation of impulses $\int N dt$

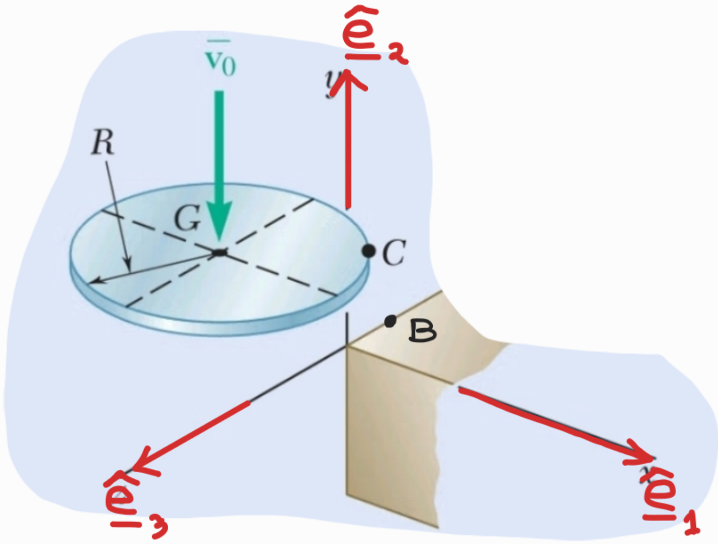
↳ $\underline{\omega}_2' \approx \underline{\omega}_2$

$\underline{v}_{c_2}' \approx \underline{v}_{c_2}$

$\Rightarrow$ Only $\underbrace{\text{6 unknowns}}$

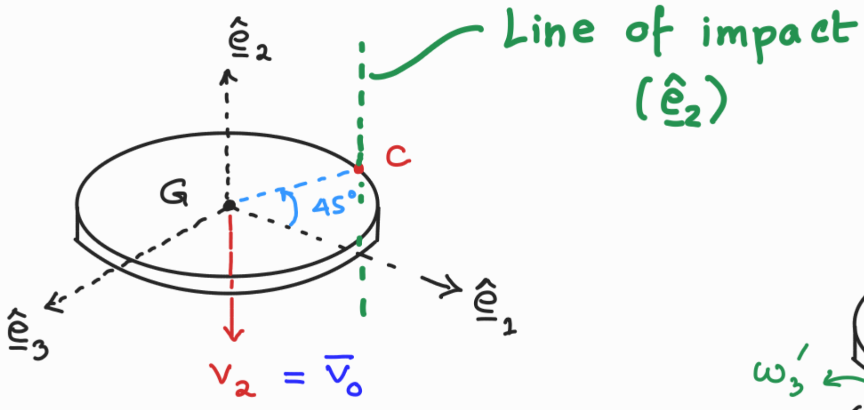
$\underline{\omega}_1'$ and $\underline{v}_{c_1}'$

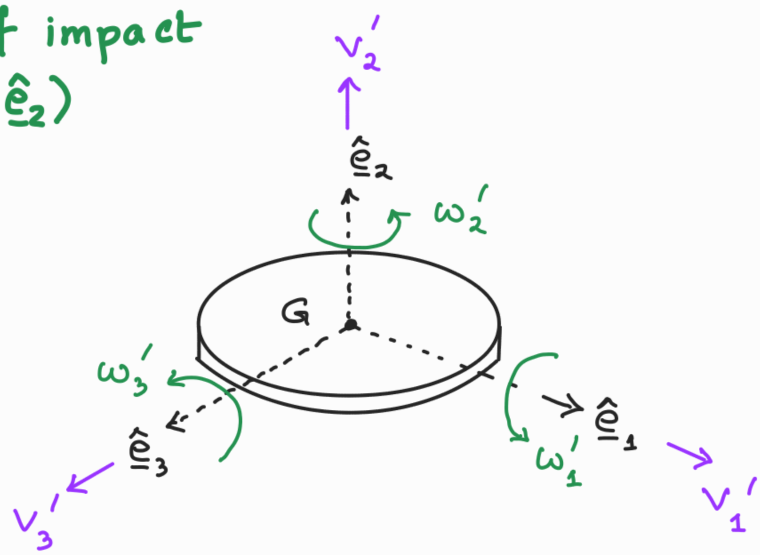
of RB ① after impact



Before impact

After impact





Knowns

$$[\underline{\omega}]_{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[\underline{v}_G]_{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}} = \begin{bmatrix} 0 \\ \bar{v}_0 \\ 0 \end{bmatrix}$$

6 unknowns

$$[\underline{\omega}']_{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}} = \begin{bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{bmatrix}$$

$$[\underline{v}_G']_{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}} = \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix}$$

The six equations (derived in Lec 20) can be used:

1) $v_{t_1} = v_{t_1}'$ & $v_{b_1} = v_{b_1}'$ (2 eqns) [Smooth collision]

2) $\underline{H}_{c_1} + \underline{r}_{c_1 O} \times m_1 \underline{v}_{c_1} = \underline{H}_{c_1}' + \underline{r}_{c_1 O} \times m_1 \underline{v}_{c_1}'$ (3 eqns)

[Conservation of angular momentum about collision pt O]

3) $e = -\dfrac{(v_{Bn} - v_{An}')}{(v_{Bn} - v_{An})}$ (1 eqn) [Relation using coeff. of restitution]

1) $v_1 = v_1'$ & $v_3 = v_3'$ (2 eqns)

Given $v_1 = 0$, and $v_3 = 0$

$$\Rightarrow v_1' = 0 \checkmark \text{ — } ① \qquad \Rightarrow v_3' = 0 \checkmark \text{ — } ②$$

$$\vec{H}_C = H_G + \vec{r}_{GC} \times m \vec{v}_{GC}$$

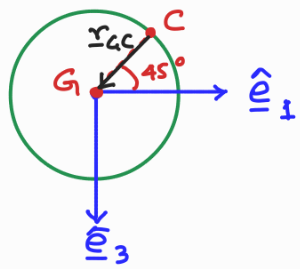
2) $$\underbrace{\underline{H}_G + \underline{r}_{GC} \times m_1 \underline{v}_G}_{\text{LHS}} = \underbrace{\underline{H}_G' + \underline{r}_{GC} \times m_1 \underline{v}_G'}_{\text{RHS}} \quad (3 \text{ eqns})$$

LHS $\Downarrow$

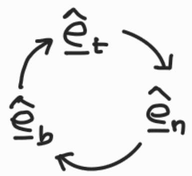
$$\underline{H}_G = \underline{\underline{I}}^G \underbrace{\underline{\omega}}_{\underline{0}\,(\text{given})} = \underline{0} \checkmark$$

$$[\underline{\underline{I}}^G]_{(e_1, e_2, e_3)} = \begin{bmatrix} \frac{m_1 r^2}{4} & 0 & 0 \\ 0 & \frac{m_1 r^2}{2} & 0 \\ 0 & 0 & \frac{m_1 r^2}{4} \end{bmatrix}$$

$$\underline{r}_{GC} = -\frac{r}{\sqrt{2}} \hat{\underline{e}}_1 + \frac{r}{\sqrt{2}} \hat{\underline{e}}_3$$



$$\underline{v}_G = -\bar{v}_0 \hat{\underline{e}}_2$$



$$\underline{r}_{GC} \times \underline{v}_G = +\frac{r\bar{v}_0}{\sqrt{2}} \hat{\underline{e}}_3 + \frac{r\bar{v}_0}{\sqrt{2}} \hat{\underline{e}}_1$$

$$\underline{H}_G + \underline{r}_{GC} \times m_1 \underline{v}_G = m_1 \left(\frac{r\bar{v}_0}{\sqrt{2}} \hat{\underline{e}}_1 + \frac{r\bar{v}_0}{\sqrt{2}} \hat{\underline{e}}_3 \right) = \begin{bmatrix} \frac{m_1 r \bar{v}_0}{\sqrt{2}} \\ 0 \\ \frac{m_1 r \bar{v}_0}{\sqrt{2}} \end{bmatrix}$$

$$\vec{V}_G' = v_2' \hat{e}_2$$

RHS : $\underline{H}_G' + \underline{r}_{GC} \times m_1 \underline{v}_G'$

$$\begin{bmatrix} \frac{m_1 r^2}{4} & 0 & 0 \\ 0 & \frac{m_1 r^2}{2} & 0 \\ 0 & 0 & \frac{m_1 r^2}{4} \end{bmatrix} \begin{bmatrix} \omega_t' \\ \omega_n' \\ \omega_b' \end{bmatrix} + m_1 \begin{bmatrix} -r/\sqrt{2} \\ 0 \\ r/\sqrt{2} \end{bmatrix} \times \begin{bmatrix} \cancelto{0}{v_1'} \\ v_2' \\ \cancelto{0}{v_3'} \end{bmatrix} \begin{matrix} (\text{from } ①) \\ \\ (\text{from } ②) \end{matrix}$$

RHS = LHS

$$\begin{bmatrix} \frac{\cancel{m_1} r^{\cancel{2}}}{4}\omega_1' - \cancel{m_1}\frac{\cancel{r} V_2'}{\sqrt{2}} \\ \frac{m_1 r^2}{2}\omega_2' \\ \frac{\cancel{m_1} r^{\cancel{2}}}{4}\omega_3' - \cancel{m_1}\frac{\cancel{r} V_2'}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\cancel{m_1}\cancel{r}\bar{V}_0}{\sqrt{2}} \\ 0 \\ \frac{\cancel{m_1}\cancel{r}\bar{V}_0}{\sqrt{2}} \end{bmatrix} \begin{matrix} — ③ \\ — ④ \\ — ⑤ \end{matrix}$$

Solving Eq ④, we get:

$$\omega_2' = 0$$

3) $\cancelto{0}{e} = -\dfrac{(\cancelto{0}{V_{B_2}'} - V_{C_2}')}{(\cancelto{0}{V_{B_2}} - V_{C_2})}$ (1 eqn)

(Plastic collision)

$V_{B_2} \equiv$ velocity of the contact point B on massive body along $\hat{e}_z$

$= 0$ (RB ② is at rest)

$$\Rightarrow V_{C_2}' = 0 \quad — ⑥$$

Using velocity transfer rule,

we can relate the velocity at G to that at C

$$\underline{V}_G' = \underline{V}_C' + \underline{\omega}' \times \underline{r}_{GC}$$ ← already calculated.

$$\Rightarrow \begin{bmatrix} 0 \\ V_2' \\ 0 \end{bmatrix} = \begin{bmatrix} ? \\ 0 \\ ? \end{bmatrix} + \begin{bmatrix} \omega_1' \\ 0 \\ \omega_3' \end{bmatrix} \times \begin{bmatrix} -r/\sqrt{2} \\ 0 \\ r/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ v_2' \\ 0 \end{bmatrix} = \begin{bmatrix} \cancel{?}\ 0 \\ 0 \\ \cancel{?}\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -(\omega_1' + \omega_3')\, r/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad v_2' = -\frac{r}{\sqrt{2}}(\omega_1' + \omega_3')$$

$$\frac{r\,\omega_1'}{4} - \frac{v_2'}{\sqrt{2}} = \frac{\bar{V}_0}{\sqrt{2}} \qquad \text{— } (3)$$

$$\frac{r\,\omega_3'}{4} - \frac{v_2'}{\sqrt{2}} = \frac{\bar{V}_0}{\sqrt{2}} \qquad \text{— } (5)$$

$$\frac{r\,\omega_1'}{4} + \frac{r}{2}(\omega_1' + \omega_3') = \frac{\bar{V}_0}{\sqrt{2}}$$

$$+$$

$$\frac{r\,\omega_3'}{4} + \frac{r}{2}(\omega_1' + \omega_3') = \frac{\bar{V}_0}{\sqrt{2}}$$

$$(\omega_1' + \omega_3')\left[\frac{r}{4} + r\right] = \sqrt{2}\,\bar{V}_0$$

$$\Rightarrow \quad (\omega_1' + \omega_3')\,\frac{5}{4}\,r = \sqrt{2}\,\bar{V}_0$$

$$\Rightarrow \quad \omega_1' + \omega_3' = \frac{4\sqrt{2}}{5}\,\frac{\bar{V}_0}{r}$$

$$v_2' = -\frac{r}{\sqrt{2}}(\omega_1' + \omega_3') = -\frac{4}{5}\,\bar{V}_0$$

Sub the above value of v_2' in Eqn (5)

$$\frac{r\omega_1'}{4} - \frac{v_2'}{\sqrt{2}} = \frac{\bar{v}_0}{\sqrt{2}}$$

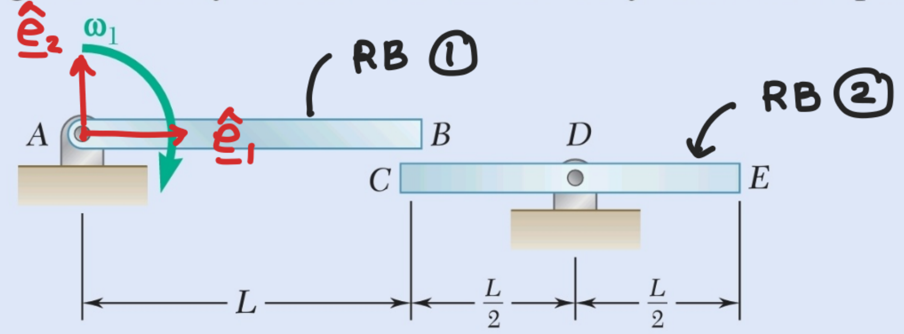
$$\Rightarrow \quad \omega_1' = \left(\frac{\bar{v}_0}{\sqrt{2}} + \frac{v_2'}{\sqrt{2}}\right)\frac{4}{r} = \frac{4}{\sqrt{2}}\frac{\bar{v}_0}{r}\left(1 - \frac{4}{5}\right)$$

$$= \frac{2\sqrt{2}}{5}\frac{\bar{v}_0}{r}$$

Similarly, you will get $\omega_2' = \dfrac{2\sqrt{2}}{5}\dfrac{\bar{v}_0}{r}$

4)

17.F6 A slender rod *CDE* of length L and mass m is attached to a pin support at its midpoint *D*. A second and identical rod *AB* is rotating about a pin support at *A* with an angular velocity $\boldsymbol{\omega}_1$ when its end *B* strikes end *C* of rod *CDE*. The coefficient of restitution between the rods is e. Draw the impulse–momentum diagrams that are needed to determine the angular velocity of each rod immediately after the impact.



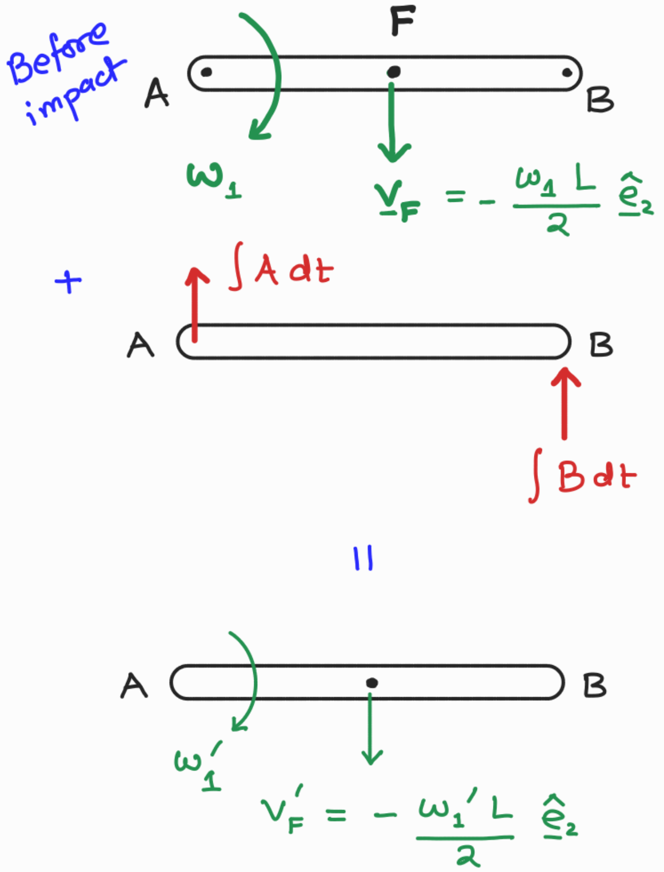
Solu : Constrained collision $\Rightarrow$ Cannot avoid calculation of unknown impulse $\int N dt$, $\int A dt$

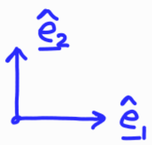
Planar 2D problem $\Rightarrow$ velocity in $\hat{e}_3$ direction is same (before & after impact.

Angular velocity vector has only one component

$$[\underline{\omega}]_{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}} = \begin{bmatrix} 0 \\ 0 \\ \checkmark \end{bmatrix}$$

Let's draw the impulse-momentum diagram for RB ①





1> Smooth collision

$$v_{F_1}' = v_{F_1} = 0 \quad \text{— ①}$$

2> Impulse-momentum along line of impact $\hat{e}_2$

$$\int F^{imp}\, dt = m_1 v_{F_2}' - m_1 v_{F_2}$$

$$\Rightarrow \int A\, dt + \int B\, dt = m_1 \left\{ v_{F_2}' - v_{F_2} \right\}$$

$$= \frac{m_1 L}{2} \left(-\omega_1' + \omega_1 \right)$$

— ②

3> Angular impulse - angular momentum abt COM of RB ①

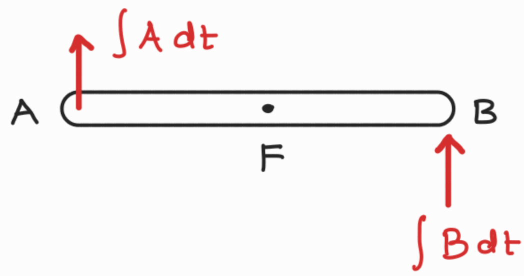
We choose the COM of RB 1 as it is a valid point for Euler's 2nd axiom

$$\circlearrowleft + \quad \int M_{F_3}^{imp}\, dt = \underbrace{H_{F_3}'}_{I_{33}^F(-\omega_1')} - \underbrace{H_{F_3}}_{I_{33}^F(-\omega_1)}$$

where H_{F_3} represents angular momentum of RB ① about F in the direction $\hat{e}_3$

$$I_{33}^F = \frac{mL^2}{12}$$

$$\Rightarrow -\left(\int A\,dt\right)\left(\frac{L}{2}\right) + \left(\int B\,dt\right)\left(\frac{L}{2}\right)$$



$$= I_{33}^{F}\left(-\omega_1'\right) - I_{33}^{F}\left(-\omega_1\right)$$

+

anticlockwise is +ve

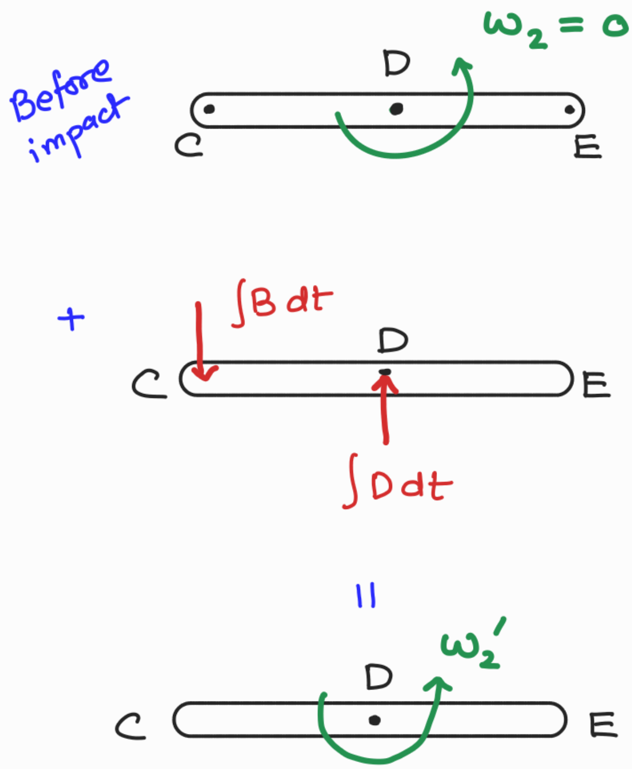
$$\Rightarrow \int B\,dt - \int A\,dt = \frac{mL}{6}\left(\omega_1 - \omega_1'\right) \quad — (3)$$

Add (2) and (3), we get:

$$\int B\,dt = \frac{mL}{3}\left(\omega_1 - \omega_1'\right)$$

Let's draw the impulse-momentum diagram for RB (2)

$\hat{e}_2$ $\hat{e}_1$



1) Point D is hinged

$$\underline{V}_D' = \underline{V}_D = \underline{0}$$

2) Impulse momentum along $\hat{e}_2$

(Not required since velocity of COM D is always zero)

3) Angular impulse- angular momentum abt point D

$$\circlearrowleft + \quad \int M\,dt = H'_{D_3} - H_{D_3}$$

$$\Rightarrow \left(\int B\,dt\right)\frac{L}{2} = I^{D}_{33}\,\omega_2' - I^{D}_{33}\,\cancel{\omega_2}^{\,0}$$

$$\Rightarrow \int B\,dt = \frac{mL}{6}\,\omega_2' \quad \text{— } (4)$$

Sub the already found value

$$\Rightarrow \frac{mL}{3}\left(\omega_1 - \omega_1'\right) = \frac{mL}{6}\,\omega_2'$$

$$\Rightarrow 2\omega_1 - 2\omega_1' = \omega_2' \quad \text{— } (A)$$

4) Use the relation of coefficient of restitution

$$e = -\frac{V'_{C_2} - V'_{B_2}}{V_{C_2} - V_{B_2}} \quad \text{along } \hat{e}_2$$

$$V_{B_2} = \omega_1 L$$

$$V'_{B_2} = \omega_1' L$$

$$V_{C_2} = \cancel{\omega_2}^{\,0}\,\frac{L}{2}$$

$$V'_{C_2} = \omega_2'\,\frac{L}{2}$$

$$\Rightarrow e = -\frac{\left(\omega_2'\frac{L}{2} - \omega_1' L\right)}{\left(0 - \omega_1 L\right)}$$

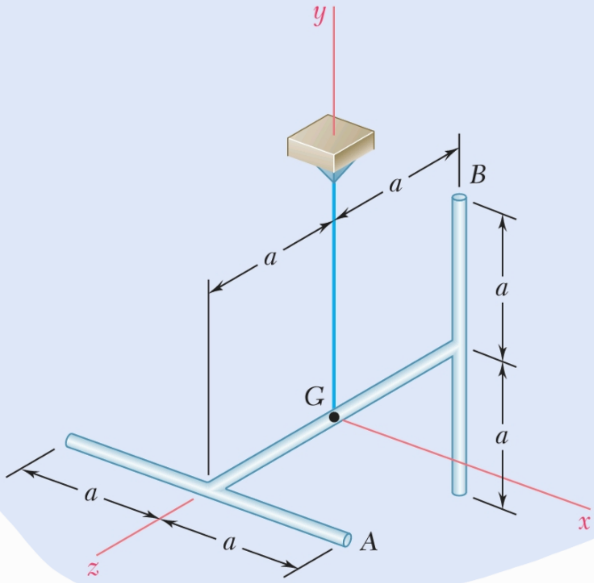
$$\Rightarrow 2\omega_1' - \omega_2' = 2e\,\omega_1 \quad \text{— } (B)$$

Use (A) & (B) to solve for ω_1' and ω_2'

Part B

1)

18.25 Three slender rods, each of mass m and length $2a$, are welded together to form the assembly shown. The assembly is hit at A in a vertical downward direction. Denoting the corresponding impulse by $\mathbf{F}\,\Delta t$, determine immediately after the impact (*a*) the velocity of the mass center G, (*b*) the angular velocity of the rod.

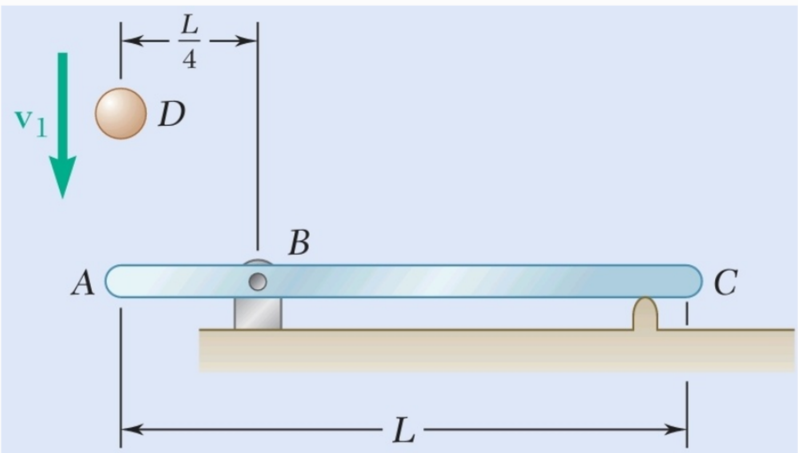


$$\omega = \frac{3F\Delta t}{8ma}(\hat{i} - 4\hat{k})$$

$$\bar{v} = 0$$

2)

17.127 and 17.128 Member ABC has a mass of 2.4 kg and is attached to a pin support at B. An 800-g sphere D strikes the end of member ABC with a vertical velocity $\mathbf{v}_1$ of 3 m/s. Knowing that $L = 750$ mm and that the coefficient of restitution between the sphere and member ABC is 0.5, determine immediately after the impact (*a*) the angular velocity of member ABC, (*b*) the velocity of the sphere.

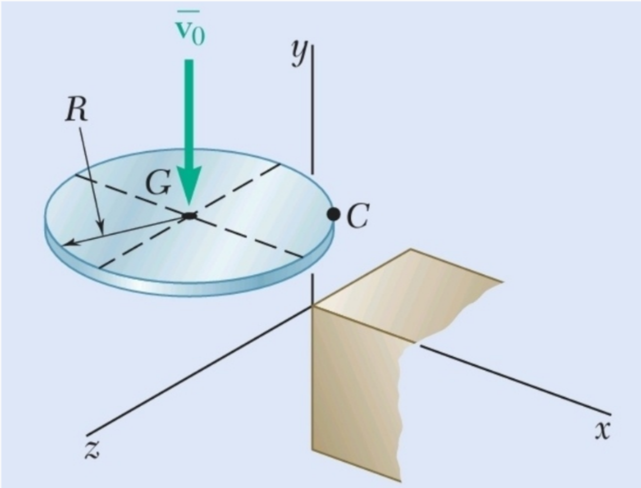


$$\omega_{ABC} = 3 \text{ rad/s} \circlearrowright$$

$$v_D = 0.938 \text{ m/s} \uparrow$$

3]

18.51 Determine the kinetic energy lost when edge C of the plate of hits the obstruction.



$$T_0 - T = \frac{1}{10} m \bar{v}_0^2$$

4]

Aa 20-*kg* uniform cylindrical roller, initially at rest, is acted upon by a 90-*N* force as shown. Knowing that the body rolls without slipping, determine (a) the velocity of its center *G* after it has moved 1.5 *m*, (b) the friction force required to prevent slipping