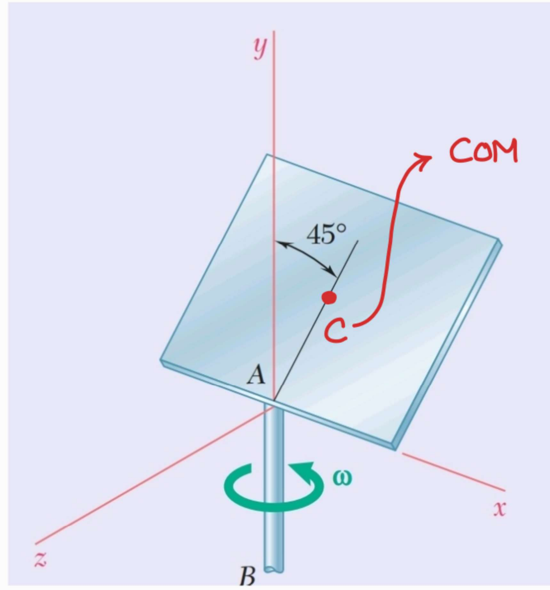
Part A solutions

1) Find kinetic energy of plate



Recall that computation of KE requires careful selection of a point on the body for which the calculation becomes easy.

Two choices for computing KE

$T|_F$

pt A has zero linear velocity, $\underline{v}_{A|F} = \underline{0}$

$\Rightarrow$ can use $T|_F = \frac{1}{2}\, \underline{\omega}_{m|F}\, \underline{\underline{I}}^{A}\, \underline{\omega}_{m|F}$

Calculation of $\underline{\underline{I}}^{A}$ needs the use of parallel axes thm to transfer from C to A

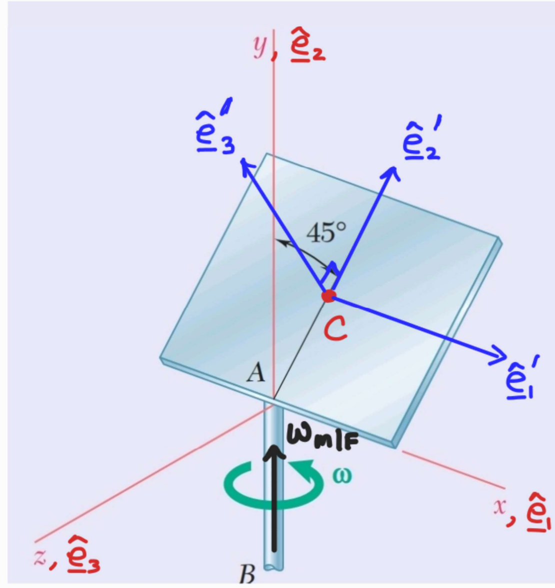
pt $C \equiv$ COM

$\Rightarrow$ use $T|_F = \frac{1}{2} m\, \underline{v}_{c|F} \cdot \underline{v}_{c|F}$

$+ \frac{1}{2}\, \underline{\omega}_{m|F}\, \underline{\underline{I}}^{c}\, \underline{\omega}_{m|F}$

$\underline{\underline{I}}_c$ can be easily compute but $\underline{v}_{c|F}$ needs to be calculated as well

For the csys given in the figure, $[\underline{\underline{I}}^c]_{(\hat{\underline{e}}_1\ \hat{\underline{e}}_2\ \hat{\underline{e}}_3)}$ is not easily calculated. as they donot coincide with the principal directions of the plate $\hat{\underline{e}}_1' - \hat{\underline{e}}_2' - \hat{\underline{e}}_3'$



$T_{|F}$ computation using C as base point

in $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ csys

$$[\underline{\omega}_{m|F}] = \begin{bmatrix} 0 \\ \omega \\ 0 \end{bmatrix}$$

$$[\underline{\underline{I}}^c] = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

Get $[\underline{\underline{I}}^c]_{(\hat{\underline{e}}_1'\ \hat{\underline{e}}_2'\ \hat{\underline{e}}_3')}$ in prin. csys $\hat{\underline{e}}_1' - \hat{\underline{e}}_2' - \hat{\underline{e}}_3'$

and then use transformation rule

$$[\underline{\underline{I}}^c]_{(\hat{\underline{e}}_1\ \hat{\underline{e}}_2\ \hat{\underline{e}}_3)} = \overset{\checkmark}{[\underline{\underline{A}}]}\ \overset{\checkmark}{[\underline{\underline{I}}^c]}_{(\hat{\underline{e}}_1'\ \hat{\underline{e}}_2'\ \hat{\underline{e}}_3')}\ \overset{\checkmark}{[\underline{\underline{A}}]^T}$$

need to calculate $[\underline{\underline{A}}]$

in principal csys $\hat{\underline{e}}_1' - \hat{\underline{e}}_2' - \hat{\underline{e}}_3'$

$$[\underline{\underline{I}}^c] = \begin{bmatrix} \frac{ma^2}{12} & 0 & 0 \\ 0 & \frac{ma^2}{12} & 0 \\ 0 & 0 & \frac{ma^2}{6} \end{bmatrix}$$

$$[\underline{\omega}_{m|F}] = \begin{bmatrix} 0 \\ \omega \cos 45° \\ \omega \sin 45° \end{bmatrix}$$

Working in the principal csys at COM is easier!

Compute $v_{C|F}$ using velocity transfer relations

$$\underline{v}_{C|F} = \cancel{\underline{v}_{A|F}}^{\,0} + \underline{\omega}_{m|F} \times \underline{r}_{CA}$$

$$= \left(\omega/\sqrt{2}\ \hat{\underline{e}}_2' + \omega/\sqrt{2}\ \hat{\underline{e}}_3' \right) \times a/2\ \hat{\underline{e}}_2'$$

$$= -a\omega/2\sqrt{2}\ \hat{\underline{e}}_1'$$

Finally compute KE using C as base pt (using $\hat{\underline{e}}_1' - \hat{\underline{e}}_2' - \hat{\underline{e}}_3'$)

$$T|_F = \frac{1}{2} m\, \underline{v}_{C|F} \cdot \underline{v}_{C|F} + \frac{1}{2} \underline{\omega}_{m|F} \cdot \underline{\underline{I}}^C \underline{\omega}_{m|F}$$

$$= \frac{1}{2} m \left[\underline{v}_{C|F}\right]^T \left[\underline{v}_{C|F}\right] + \frac{1}{2} \left[\underline{\omega}_{m|F}\right] \left[\underline{\underline{I}}^C\right] \left[\underline{\omega}_{m|F}\right]$$

$$= \frac{m}{2} \cdot \frac{a^2\omega^2}{8} + \frac{1}{2} \begin{bmatrix} 0 \\ \omega/\sqrt{2} \\ \omega/\sqrt{2} \end{bmatrix}^T \begin{bmatrix} \frac{ma^2}{12} & 0 & 0 \\ 0 & \frac{ma^2}{12} & 0 \\ 0 & 0 & \frac{ma^2}{6} \end{bmatrix} \begin{bmatrix} 0 \\ \omega/\sqrt{2} \\ \omega/\sqrt{2} \end{bmatrix}$$

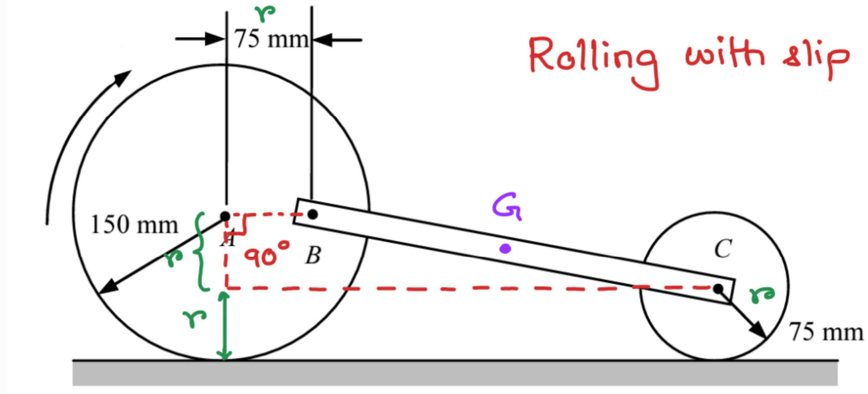
$$= \frac{ma^2\omega^2}{16} + \frac{1}{2} \left[\frac{\omega^2}{2} \cdot \frac{ma^2}{12} + \frac{\omega^2}{2} \cdot \frac{ma^2}{6} \right]$$

$$= \frac{ma^2\omega^2}{16} + \frac{3ma^2\omega^2}{48}$$

$$= \frac{ma^2\omega^2}{8}$$

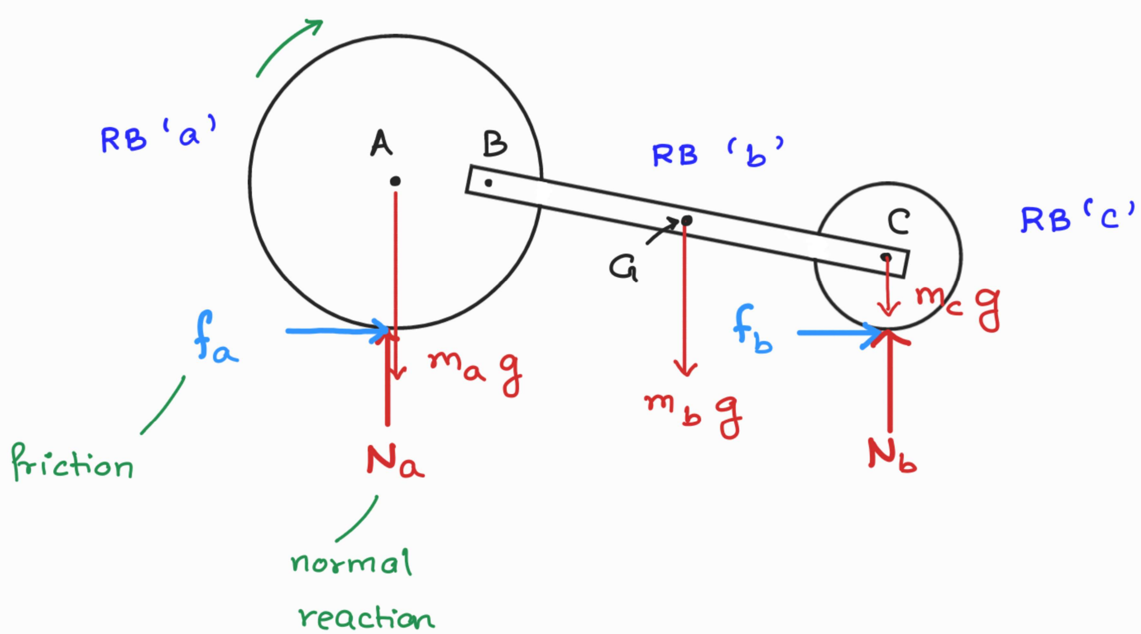
$$\boxed{T|_F = \frac{ma^2\omega^2}{8}}$$

2) Determine velocity of rod BC after disk has rotated by 90°



- Since there are Two positions of rod BC : initial – 1 and final – 2 , and we are interested in knowing the velocity of rod, we use the work-energy principle.

- Consider system = rod + two disks , and draw FBD for the system The work done by internal forces are equal and opposite and hence cancel out. We therefore consider only forces/moments external to the "system"



Work energy principle $\Rightarrow$ $\underset{\sim\sim\sim}{W_{1\to 2}} = \Delta T = T_{2|F} - T_{1|F}$

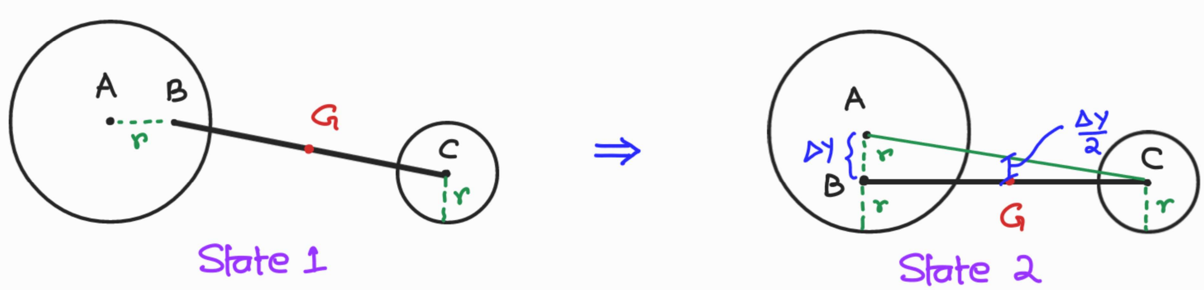
Work done by ext. forces in moving the body from state $1 \to 2$

1) Frictional forces f_a and f_b do zero work as velocities at the no-slip contact points are zero

2) The normal reactions N_a and N_b are a part of reaction force system, and reaction force systems do not any work as they arise due to constraints in motion.

3) The forces that do non-zero work in this problem are the weights due to gravity (which are constant forces, hence also conservative forces) : $m_a g$, $m_b g$, and $m_c g$

Work done by weights due to gravity



Work done $W_{1\to 2}$ $= m_b g \cdot \frac{\Delta y}{2}$ ← vertical disp of G going from state 1 to state 2

$= m_b g \cdot \frac{r}{2}$

According to work-energy principle,

$$\boxed{W_{1\to 2} = \Delta T} = T_2 - \boxed{T_1 \text{ — KE at state 1}}$$

T_2 — KE at state 2

$$\Rightarrow m_b\, g \cdot \frac{r}{2} = (T^a + T^b + T^c)_2 - \cancel{(T^a + T^b + T^c)_1}^{\,0}$$

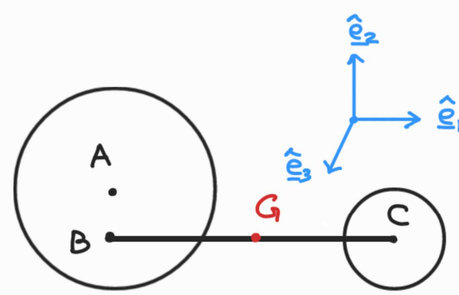
("system" was at rest initially)

$$\Rightarrow (T^a + T^b + T^c)_2 = m_b\, g \cdot \frac{r}{2} \quad \text{— } (**)$$

Kinetic energy of the system at state 2

$(T^a)_2 = ?$ $\qquad (T^b)_2 = ?$ $\qquad (T^c)_2 = ?$

Using $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$ csys for calculation



State 2

$$(T^a)_2 = \frac{1}{2} m_a [\underline{v}_{A|I}]^T [\underline{v}_{A|I}] + \frac{1}{2} [\underline{\omega}_{a|I}]^T [\underline{\underline{I}}^A] [\underline{\omega}_{a|I}]$$

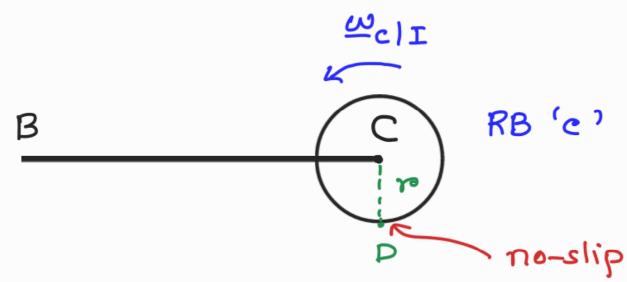
$$(T^b)_2 = \frac{1}{2} m_b [\underline{v}_{G|I}]^T [\underline{v}_{G|I}] + \frac{1}{2} [\underline{\omega}_{b|I}]^T [\underline{\underline{I}}^G] [\underline{\omega}_{b|I}]$$

$$(T^c)_2 = \frac{1}{2} m [\underline{v}_{c|I}]^T [\underline{v}_{c|I}] + \frac{1}{2} [\underline{\omega}_{c|I}]^T [\underline{\underline{I}}^c] [\underline{\omega}_{c|I}]$$

Use kinematics to relate $\underline{v}_{G|I}$, $\underline{v}_{A|I}$, and $\underline{v}_{c|I}$

Compute $\underline{v}_{c|I}$

$$\overset{v_c \hat{e}_1}{\underline{v}_{c|I}} = \cancelto{0}{\underline{v}_{D|I}} + \overset{\omega_c \hat{e}_3}{\underline{\omega}_{c|I}} \times \overset{r \hat{e}_2}{\underline{r}_{CD}}$$

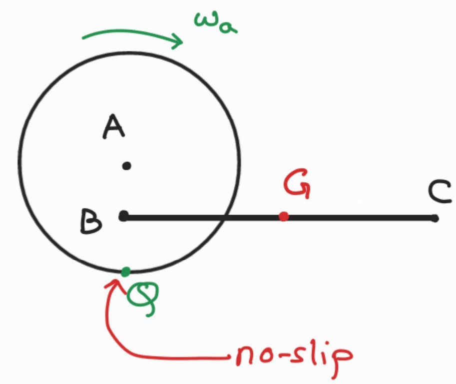


$$\Rightarrow \underline{v}_{c|I} = \omega_c r \,\hat{e}_1$$

Compute $\underline{v}_{A|I}$

$$\underline{v}_{A|I} = \overset{0}{\underline{v}_{Q|I}} + \overset{-\omega_a \hat{e}_3}{\underline{\omega}_{a|I}} \times \overset{2r \hat{e}_2}{\underline{r}_{AQ}}$$

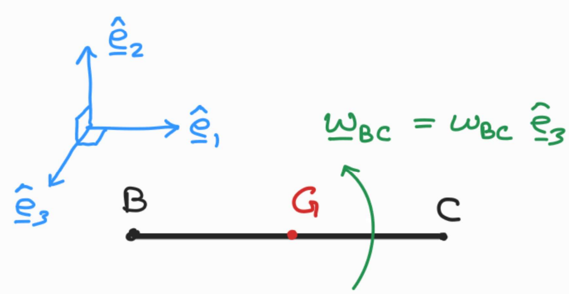
$$= 2\omega_a r \,\hat{e}_1$$



Compute $\underline{v}_{B|I}$

$$\underline{v}_{B|I} = \overset{0}{\underline{v}_{Q|I}} + \overset{-\omega_a \hat{e}_3}{\underline{\omega}_{a|I}} \times \overset{r \hat{e}_2}{\underline{r}_{AQ}}$$

$$= \omega_a r \,\hat{e}_1$$



Compute $\underline{\omega}_{BC}$

$$\underline{v}_{B|I} = \underline{v}_{c|I} + \underline{\omega}_{BC} \times \underline{r}_{BC}$$

$$\Rightarrow \omega_a r \,\hat{e}_1 = \omega_c r \,\hat{e}_1 + \omega_{BC}\,\hat{e}_3 \times l\,\hat{e}_1$$

$$\Rightarrow \omega_a r \,\hat{e}_1 = \omega_c r \,\hat{e}_1 + \omega_{BC}\, l \,\hat{e}_2$$

$\Rightarrow$ $\omega_{BC} = 0$ ($\hat{e}_2$-comp)

& $\omega_a = \omega_c$ ($\hat{e}_1$-comp)

Compute $\underline{v}_{G|I}$

$$\underline{v}_{G|I} = \underline{v}_{c|I} + \cancelto{0}{\underline{\omega}_{BC}} \times \underline{r}_{GC} = \omega_a r \,\hat{e}_1$$

Compute kinetic energies of the individual RBs 'a', 'b', 'c'

$$\left(T^{a}_{|I}\right)_2 = \frac{1}{2} m_a \left[\underline{v}_{A|I}\right]^T \left[\underline{v}_{A|I}\right] + \frac{1}{2} \left[\underline{\omega}_{a|I}\right]^T \left[\underline{\underline{I}}^A\right] \left[\underline{\omega}_{a|I}\right]$$

$$= \frac{1}{2} m_a \begin{bmatrix} 2r\omega_a \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 2r\omega_a \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ -\omega_a \end{bmatrix}^T \begin{bmatrix} \checkmark & 0 & 0 \\ 0 & \checkmark & 0 \\ 0 & 0 & \frac{m_a R^2}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -\omega_a \end{bmatrix}$$

$$= \frac{1}{2} m_a \cdot (2r\omega_a)^2 + \frac{1}{2} (\omega_a)^2 \cdot \left(\frac{m_a (2r)^2}{2}\right) \qquad [R = 2r]$$

$$= \frac{m_a \; 4 r^2 \omega_a^2}{2} + \frac{m_a \; 4 r^2 \omega_a^2}{4}$$

$$= 3 m_a r^2 \omega_a^2$$

$$\left(T^{b}_{|I}\right)_2 = \frac{1}{2} m_b \left[\underline{v}_{G|I}\right]^T \left[\underline{v}_{G|I}\right] + \frac{1}{2} \left[\underline{\omega}_{b|I}\right]^T \left[\underline{\underline{I}}^G\right] \left[\underline{\omega}_{b|I}\right]$$

$$= \frac{1}{2} m_b \begin{bmatrix} r\omega_a \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} r\omega_a \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \checkmark & 0 & 0 \\ 0 & \checkmark & 0 \\ 0 & 0 & \checkmark \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} m_b r^2 \omega_a^2$$

$$\left(T^{c}_{|I}\right)_2 = \frac{1}{2} m_c \left[\underline{v}_{C|I}\right]^T \left[\underline{v}_{C|I}\right] + \frac{1}{2} \left[\underline{\omega}_{c|I}\right]^T \left[\underline{\underline{I}}\right] \left[\underline{\omega}_{c|I}\right]$$

$$= \frac{1}{2} m_c \begin{bmatrix} r\omega_a \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} r\omega_a \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \omega_c \end{bmatrix}^T \begin{bmatrix} \checkmark & 0 & 0 \\ 0 & \checkmark & 0 \\ 0 & 0 & \frac{m_c r^2}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega_c \end{bmatrix}$$

$$= \frac{1}{2} m_c r^2 \omega_a^2 + \frac{1}{2} \omega_a^2 \frac{m_c r^2}{2} \qquad [\omega_c = \omega_a]$$

$$= \frac{3}{4} m_c r^2 \omega_a^2$$

Adding $(T_{1I}^a)_2$, $(T_{1I}^b)_2$, and $(T_{1I}^c)_2$ for KE of system

$$(T_{1I}^a + T_{1I}^b + T_{1I}^c)_2 = 3 \overset{6}{m_a} r^2 \omega_a^2 + \frac{1}{2} \overset{5}{m_b} r^2 \omega_a^2 + \frac{3}{4} \overset{1.5}{m_c} r^2 \omega_a^2$$

$$[v_B = \omega_a r]$$

$$= 18 v_B^2 + \frac{5}{2} v_B^2 + \frac{9}{8} v_B^2$$

$$= 21.6 v_B^2$$

Using work-energy relation (**)

$$21.6 v_B^2 = \overset{5}{m_b} g \frac{\overset{0.075}{r}}{2} \qquad g = 9.81$$

$$\Rightarrow v_B = 0.29 \text{ m/s}$$