Tutorial 1

1> **Theme of the problem: Finding $\underline{v}_{P|F}$ and $\underline{a}_{P|F}$ using a frame-fixed Cartesian CSYS**. A pin P moves in a fixed parabolic slot whose equation is given by, $x = cy^2$, and in a straight horizontal slot as shown in Fig. 1. The straight slot translates in the y-direction at a constant acceleration a_0, starting from rest at t=0, when the pin is at the origin. Find the position, velocity, and acceleration of P at time t.

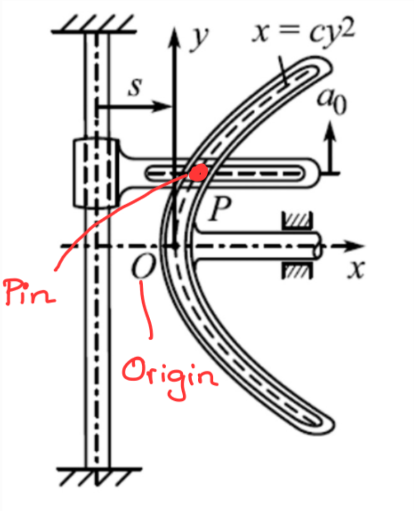


Fig 1

- Two-dimensional problem $\Rightarrow$ 2 components of any vector

- $\underline{r}_{P|O} = x_p \hat{i} + y_p \hat{j}$

  $\underline{v}_{P|F} = v_x \hat{i} + v_y \hat{j}$

  $\underline{a}_{P|F} = a_x \hat{i} + a_y \hat{j}$

- The origin is at O
- Pin P starts its motion from O at $t=0$

  $\Rightarrow \quad y_p = 0 \quad$ at $t = 0$

- Since pin P is constrained to move along the slots, the vertical acceleration of pin P $= a_o$

$$a_y = \ddot{y}_p \quad \text{(in Cartesian csys)}$$
$$= a_o \quad \text{(due to kinematic constraint)}$$

$$\Rightarrow \quad \ddot{y}_p(t) = a_o \leftarrow a_y$$

$$\Rightarrow \quad \dot{y}_p(t) = \int_0^t a_o \, dt = a_o t \leftarrow v_y$$

$$\Rightarrow \quad y_p(t) = \int_0^t a_o t \, dt = \frac{1}{2} a_o t^2 \leftarrow y(t)$$

For the components along x-direction, we will use the relation

$x = c y^2$

$$x_p(t) = c\, y_p^2(t)$$ sub in the value of $y(t)$

$$= \frac{c}{4} a^2 t^4$$

$$\Rightarrow \dot{x}_p(t) = \frac{c}{4} a^2\, 4 t^3 = c a^2 t^3 \leftarrow v_x$$

$$\Rightarrow \ddot{x}_p(t) = 3 c a^2 t^2 \leftarrow a_x$$

Therefore,

$$\underline{r}_{P|O} = \frac{c}{4} a^2 t^4\, \hat{i} + \frac{1}{2} a_o t^2\, \hat{j}$$

$$\underline{v}_{P|F} = c a^2 t^3\, \hat{i} + a_o t\, \hat{j}$$

$$\underline{a}_{P|F} = 3 c a^2 t^2\, \hat{i} + a_o\, \hat{j}$$

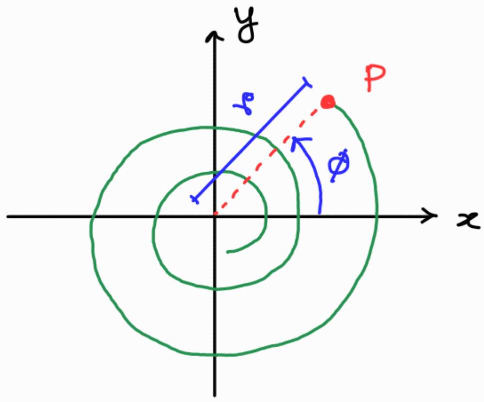
2) **Theme of the problem: Finding $\underline{v}_{P|F}$ and $\underline{a}_{P|F}$ using the cylindrical-polar CSYS.** A particle moves along a logarithmic spiral in the $x-y$ plane, given by the equation, $r = ce^{\phi}$, where c is a constant. If $\dot{\phi} = \frac{b}{r^2}$, where, $b > 0$ and is a constant, find the velocity and the acceleration of the particle when the coordinates of its location are (r, θ). Also, find the radius of curvature (ρ) of its trajectory at this location. Note the radius of curvature ρ, may be obtained as $\rho = \frac{|\underline{v}_{P|F}|^3}{|\underline{v}_{P|F} \times \underline{a}_{P|F}|}$ (this expression will be derived in one of the lecture sessions soon).

In reality (or exams), you may not be explicitly told to find velocity / acceleration in Cylindrical csys. Rather a judgement must be made by you which csys is best suited for analyzing the prob.

In this case, the cylindrical csys seems an appropriate choice since the $\underline{e}_z$-axis is fixed at a center and the motion of particle is have a circular-like pattern around the center.

- Given trajectory : $r = c e^{\phi}$, $\dot{\phi} = \frac{b}{r^2}$

- Planar problem $\Rightarrow$ $\hat{\underline{e}}_r - \hat{\underline{e}}_\phi$ ($\hat{\underline{e}}_z$ not reqd.)



Recall in cylindrical csys

	$\hat{\underline{e}}_r$	$\hat{\underline{e}}_\phi$	~~$\hat{\underline{e}}_z$~~
$\underline{r}_{P\vert F}$	r	0	~~z~~
$\underline{v}_{P\vert F}$	$\dot{r}$	$r\dot{\phi}$	~~$\dot{z}$~~
$\underline{a}_{P\vert F}$	$\ddot{r} - r\dot{\phi}^2$	$r\ddot{\phi} + 2\dot{r}\dot{\phi}$	~~$\ddot{z}$~~

* We will only need $\hat{\underline{e}}_r$ and $\hat{\underline{e}}_\phi$ components

$$\dot{r} = \frac{d}{dt}\left(c e^{\phi}\right) = c e^{\phi}\,\dot{\phi} = r\dot{\phi}$$

$$r\dot{\phi} = \cancel{r}\; b/\cancel{r^2} = b/r = b/c e^{\phi}$$

$$\ddot{r} = \frac{d}{dt}\left(r\dot{\phi}\right) = \frac{d}{dt}\left(b/r\right) = -b/r^2\; \dot{r} = -b/r^2\left(r\dot{\phi}\right) = -b^2/r^3$$

$$r\dot{\phi}^2 = \left(r\dot{\phi}\right)\dot{\phi} = b/r \cdot b/r^2 = b^2/r^3$$

$$r\ddot{\phi} = r\,\frac{d}{dt}\left(b/r^2\right) = r\left(-2b/r^3\;\dot{r}\right) = \cancel{r}\left(-2b/r^{\cancel{3}\,2}\right)\left(r\dot{\phi}\right)$$

$$= -2b/r^2 \cdot b/r = -2b^2/r^3$$

$$\dot{r}\dot{\phi} = \left(r\dot{\phi}\right)\dot{\phi} = b^2/r^3$$

$$\underline{v}_{P\vert F} = \dot{r}\,\hat{\underline{e}}_r + r\dot{\phi}\,\hat{\underline{e}}_\phi$$

$$= \frac{b}{r}\left(\hat{\underline{e}}_r + \hat{\underline{e}}_\phi\right)$$

$$\underline{a}_{P|F} = (\ddot{r} - r\dot{\phi}^2)\,\hat{\underline{e}}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\hat{\underline{e}}_\phi$$

$$= \left(-b^2/r^3 - b^2/r^3\right)\hat{\underline{e}}_r + \left(\cancel{-2b^2/r^3} + \cancel{2b^2/r^3}\right)\hat{\underline{e}}_\phi$$

$$= -\frac{2b^2}{r^3}\,\hat{\underline{e}}_r$$

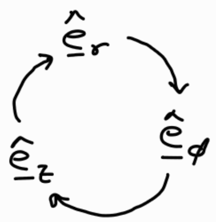
The radius of curvature ρ of the path can be obtained as:

$$\rho = \frac{|\underline{v}_{P|F}|^3}{|\underline{v}_{P|F} \times \underline{a}_{P|F}|}$$

Now,
$$\underline{v}_{P|F} \times \underline{a}_{P|F} = -\frac{2b^2}{r^4}\cdot b\,(\hat{\underline{e}}_r + \hat{\underline{e}}_\phi) \times \hat{\underline{e}}_r$$

$$= \frac{2b^3}{r^4}\,\hat{\underline{e}}_z$$

normal to the
x-y plane



$\hat{\underline{e}}_\phi \times \hat{\underline{e}}_r = -\hat{\underline{e}}_z$

$$|\underline{v}_{P|F} \times \underline{a}_{P|F}| = 2b^3/r^4$$

$$|v_{P|F}| = \sqrt{2}\, b/r$$

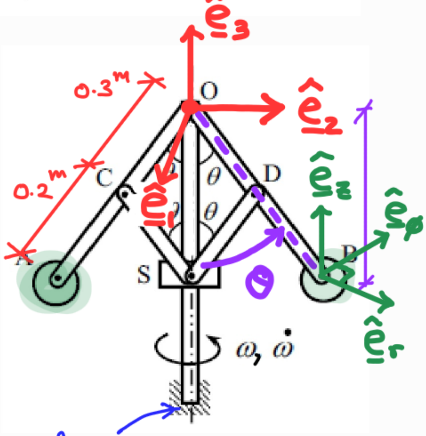
$$\rho = \frac{(\sqrt{2})^3\,(b/r)^3}{2b^3/r^4} = \sqrt{2}\, r$$

3> **Theme of the problem: Finding $\underline{v}_{P|F}$ and $\underline{a}_{P|F}$ using the cylindrical-polar CSYS**
Consider the fly-ball governor shown in the Fig. 2. Arms OA and OB are hinged to the shaft at O, however, when the sleeve, S, moves up, the angle θ increases, and the balls move radially outward and upward. At this instant, $\theta = 40°$; $\omega = 2\text{rad/s}$; $\dot{\omega} = 0.4\text{rad/s}^2$ and the sleeve, S is moving up at a speed $v = 2\text{m/s}$ which is changing at the rate $a = -0.1\text{m/s}^2$ (w.r.t. the ground frame). OA $= 0.5$m and OC $= 0.3$m.
Find the velocity and acceleration of ball B, with respect to ground, at this instant.

ref. frame

$\underline{v}_{B|F}$? $\underline{a}_{B|F}$?

Ball



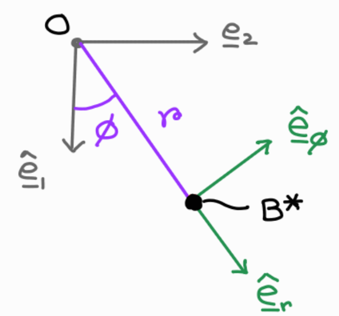
fixed frame F at ground.

- Ground is the reference frame
- Since the rotating motion is happening abt the OS axis, we make the use of cylindrical csys (r, ϕ, z) with origin fixed at O and $\hat{e}_z$-direction along OS
- Given: Speed and acceleration of sleeve S

$$\underline{v}_{S|F} = 2\,\hat{e}_z$$
$$\underline{a}_{S|F} = -0.1\,\text{m/s}^2\,\hat{e}_z$$

Angle at given time instant $\{\ \theta = 40°$

Angular speed and acc. $\begin{cases} \omega = 2\ \text{rad/s} = \dot{\phi} \\ \dot{\omega} = 0.4\ \text{rad/s}^2 = \ddot{\phi} \end{cases}$



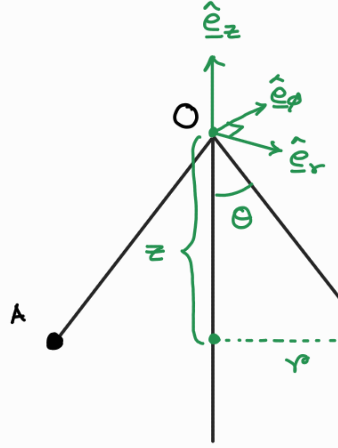
B* is the proj. of B on $\hat{e}_1 - \hat{e}_2$ plane

At any instant, the velocity & acc. of the ball B is given by:

$$\underline{v}_{B|F} = \dot{r}\,\hat{e}_r + r\overset{\omega}{\dot{\phi}}\,\hat{e}_\phi + \dot{z}\,\hat{e}_z$$

$$\underline{a}_{B|F} = (\ddot{r} - r\overset{\omega}{\dot{\phi}}{}^2)\,\hat{e}_r + (r\overset{\dot{\omega}}{\ddot{\phi}} + 2\dot{r}\overset{\omega}{\dot{\phi}})\,\hat{e}_\phi + \ddot{z}\,\hat{e}_z$$

We need to calculate the value of 'r' (OB*), $\dot{r}$, $\ddot{r}$, and $z, \dot{z}, \ddot{z}$!



$$r = OB^* = OB\sin\theta = 0.5\sin\theta$$

θ is changing with time

$$\dot{r} = \frac{d}{dt}(0.5\sin\theta) = 0.5\cos\theta \cdot \dot{\theta}$$

unknowns

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = -0.5\sin\theta \cdot \dot{\theta}^2 + 0.5\cos\theta\,\ddot{\theta}$$

Similarly,

$$z = -OB\cos\theta = -0.5\cos\theta$$

$$\dot{z} = 0.5\sin\theta \cdot \dot{\theta}, \qquad \ddot{z} = 0.5\cos\theta \cdot \dot{\theta}^2 + \underbrace{0.5\sin\theta \cdot \ddot{\theta}}_{}$$

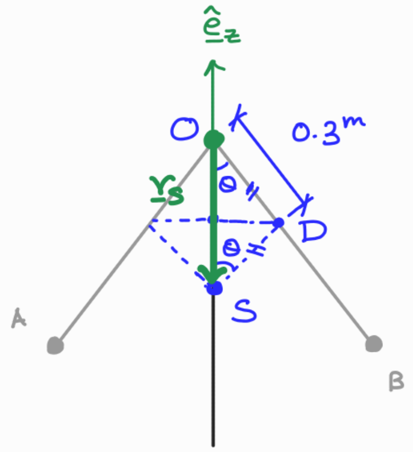
unknowns

To find $\dot{\theta}$ and $\ddot{\theta}$ at time t, we compute the vel. & acc. of sleeve S in cylindrical csys and relate it to the values given.

$$\because \quad OD = DS$$

$$\Rightarrow \underline{r}_s = -2\,OD\cos\theta\;\hat{e}_z$$

$$= -0.6\cos\theta\;\hat{e}_z$$



Velocity of sleeve S : $\underline{v}_{S|F} = \dfrac{d}{dt}\{\underline{r}_s\}\Big|_G$

$$\Rightarrow 2\,m/s\;\hat{e}_z = 0.6\,m\;\sin\theta \cdot \dot{\theta}\;\hat{e}_z \qquad [\hat{e}_z \text{ is time-invariant}]$$

$$\Rightarrow \dot{\theta} = \frac{2}{0.6\sin\theta}$$

At the current instant, $\theta = 40°$

$$\Rightarrow \boxed{\dot{\theta} = \frac{2}{0.6\sin 40°} = 5.19\;rad/s}$$

units are imp

Acceleration of sleeve S : $\underline{a}_{S|F} = \frac{d}{dt}\left\{ \underline{v}_{S|F} \right\}\Big|_F$

$$= \frac{d}{dt}\left\{ 0.6 \sin\theta \, \dot{\theta} \, \hat{\underline{e}}_z \right\}\Big|_F$$

$$\Rightarrow -0.1 \, \hat{\underline{e}}_z = \left(0.6 \cos\theta \, \dot{\theta}^2 + 0.6 \sin\theta . \ddot{\theta} \right) \hat{\underline{e}}_z$$

$$\Rightarrow -0.1 = 0.6 \cos\theta \, \dot{\theta}^2 + 0.6 \sin\theta . \ddot{\theta}$$

At the current instant, $\theta = 40°$, $\dot{\theta} = 5.19$ rad/s

$$\Rightarrow \boxed{\ddot{\theta} = \frac{-0.1 - 0.6 \cos\theta \, \dot{\theta}^2}{0.6 \sin\theta} = -32.36 \, \underbrace{\text{rad/s}^2}_{\text{units are imp!}}}$$

With the information of θ, $\dot{\theta}$, and $\ddot{\theta}$, we can now find r, $\dot{r}$, $\ddot{r}$, z, $\dot{z}$, and $\ddot{z}$, and finally compute $\underline{v}_{B|F}$ and $\underline{a}_{B|F}$ at the current time instant

$$\underline{v}_{B|F} = \dot{r} \, \hat{\underline{e}}_r + r\omega \, \hat{\underline{e}}_\phi + \dot{z} \, \hat{\underline{e}}_z \qquad [\theta = 40°]$$

$$= (0.5 \cos\theta)(5.19) \, \hat{\underline{e}}_r + (0.5 \sin\theta)(2) \, \hat{\underline{e}}_\phi + (0.5 \sin\theta)(5.19) \, \hat{\underline{e}}_z$$

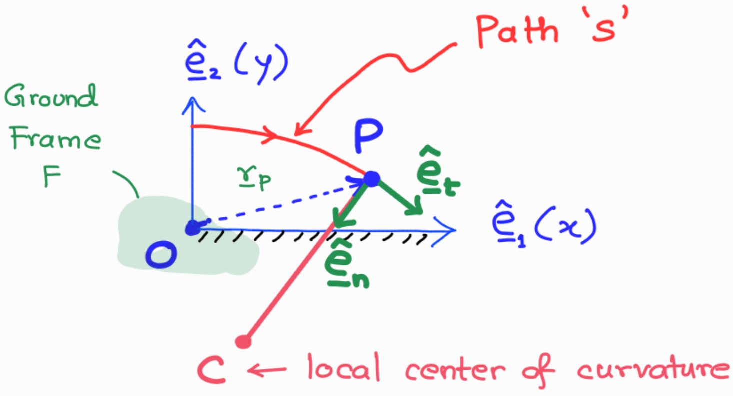
$$= 1.98626 \, \hat{\underline{e}}_r + 0.64279 \, \hat{\underline{e}}_\phi + 1.66667 \, \hat{\underline{e}}_z \ (\text{m/s})$$

$$\underline{a}_{B|F} = a_r \, \underline{e}_r + a_\phi \, \underline{e}_\phi + a_z \, \underline{e}_z$$

$$a_r = \ddot{r} - r\omega^2 = \left(0.5 \cos\theta \, \ddot{\theta} - 0.5 \sin\theta . \dot{\theta}^2\right) - r\omega^2 = -22.3031 \text{ m/s}^2$$

$$a_\phi = r\ddot{\phi} + 2\dot{r}\dot{\phi} = 8.0736 \text{ m/s}^2, \qquad a_z = \ddot{z} = -0.0833242 \text{ m/s}^2$$

4) **Theme of the problem: Finding $\underline{v}_{P|F}$ and $\underline{a}_{P|F}$ using the path CSYS** A small block of mass, $m = 0.5$kg, slides down a hill whose shape may be approximated by $y = H\cos(\pi x/L)$, where $H = 200$m and $L = 800$m. Its speed at the position shown in Fig. 3 is 40m/s. Find the rate of increase of speed in this position if the coefficient of friction between the hill and the block is 0.2.



$$\underline{r}_P = x_P\,\hat{\underline{e}}_1 + y_P\,\hat{\underline{e}}_2$$

$$x_P = 3L/8\ , \quad y_P = H\cos\left(\frac{3\pi}{8}\right) = 0.383\,H$$

[turn radians 'mode' on in Calc]

For a trajectory confined to x-y plane of a body-fixed Cartesian csys $O-\hat{\underline{e}}_1-\hat{\underline{e}}_2-\hat{\underline{e}}_3$ s.t. $\underline{r}_{P|F} = x\,\hat{\underline{e}}_1 + f(x)\,\hat{\underline{e}}_2$, then

$$\boxed{\rho = \frac{\left[1 + \left(\frac{df}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2f}{dx^2}\right|}}$$

Here, $y = f(x) = H\cos\left(\frac{\pi x}{L}\right)$

in radians

$$\left.\frac{dy}{dx}\right|_{x=\frac{3L}{8}} = \left.\frac{df}{dx}\right|_{x=\frac{3L}{8}} = \left.-H\frac{\pi}{L}\sin\left(\frac{\pi x}{L}\right)\right|_{x=\frac{3L}{8}} = -0.7256$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{3L}{8}} = \left.\frac{d^2f}{dx^2}\right|_{x=\frac{3L}{8}} = \left.-H\frac{\pi^2}{L^2}\cos\left(\frac{\pi x}{L}\right)\right|_{x=\frac{3L}{8}} = -0.00118$$

Using the formula for local radius of curvature ρ,

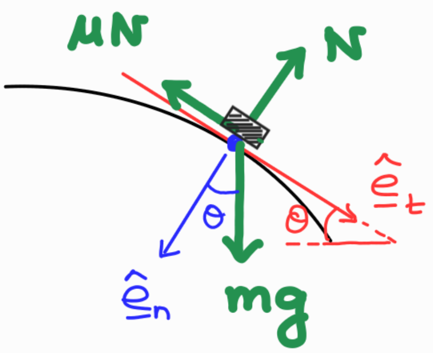
$$\rho = \frac{\left[1 + (-0.7256)^2\right]^{3/2}}{|-0.001180|} = 1597.92 \text{ m}$$

$$\underline{v}_{P|F} = \underbrace{\dot{s}}_{= 40\text{m/s (Given)}} \,\hat{\underline{e}}_t \quad \text{and} \quad \underline{a}_{P|F} = \underbrace{\underline{\underline{\ddot{s}}}}_{a_t} \hat{\underline{e}}_t + \underbrace{\frac{\dot{s}^2}{\rho}}_{a_n} \hat{\underline{e}}_n$$

a_n (known)

Unknown

Obtain $\ddot{s}$ from balance of forces of the FBD of block



* Prior to that, calculate θ

$$\tan\theta\Big|_{x=\frac{3L}{8}} = \left|\frac{dy}{dx}\Big|_{x=\frac{3L}{8}}\right| = 0.7256$$

$$\Downarrow$$

$$\sin\theta\Big|_{x=\frac{3L}{8}} = 0.5873$$

$$\cos\theta\Big|_{x=\frac{3L}{8}} = 0.8094$$

$$\underline{F}^{ext}_{net} = m\,\underline{a}_{P|F}$$

Force balance in normal direction $\hat{\underline{e}}_n$

$$\sum F_n = m a_n \qquad a_n = \frac{\dot{s}^2}{\rho} = \frac{40^2}{1597.92} = 1.001$$

$$\Rightarrow -N + mg\cos\theta = m a_n \quad \dot{s}^2/\rho$$

$$\Rightarrow N = m\left(g\cos\theta - \frac{\dot{s}^2}{\rho}\right) = 0.5\left[9.81 \cdot 0.8094 - \frac{40^2}{1597.92}\right]$$

$$= 3.469 \text{ Newton}$$

Force balance in the tangential direction $\hat{\underline{e}}_t$

$$\sum F_t = m a_t$$

$$\Rightarrow \quad mg\sin\theta - \overset{0.2\text{ (Given)}}{\mu} N = m\ddot{s}$$

$$\Rightarrow \quad \ddot{s} = \frac{mg\sin\theta - \mu N}{m} = \frac{\left[0.5 \cdot 9.81 \cdot 0.5873 - 0.2 \cdot 3.469\right]}{0.5}$$

$$= 4.374 \ m/s^2$$

In path csys

$$\therefore \quad \underline{v}_{P|F} = 40\ \hat{\underline{e}}_t \ (m/s)$$

$$\underline{a}_{P|F} = 4.374\ \hat{\underline{e}}_t + 1.001\ \hat{\underline{e}}_n \ (m/s^2)$$