Recap

In an attempt to reduce a set of forces and couples acting on a system, we talked about equivalent force systems. Especially, we are interested in finding out a resultant force $\underline{F}_R$ and a resultant moment about a point A, $\underline{M}_A$.

$$\underline{F}_R = \sum_{i=1}^{N} \underline{F}_i$$

$$\underline{M}_o = \sum_{i=1}^{N} \underline{r}_{iA} \times \underline{F}_i + \sum_{j=1}^{T} \underline{C}_j$$

We also introduced the concept of wrench, where the resultant moment is directed along the resultant force $\underline{F}_R$.

We then looked into finding resultant force and moments for special cases: coplanar force systems, parallel force system

In this lecture, we will start with finding the center of parallel force systems. Then, we will look into coplanar & parallel distributed force systems.

Later, we will introduce a variety of other topics on interaction of forces between two bodies and FBDs.

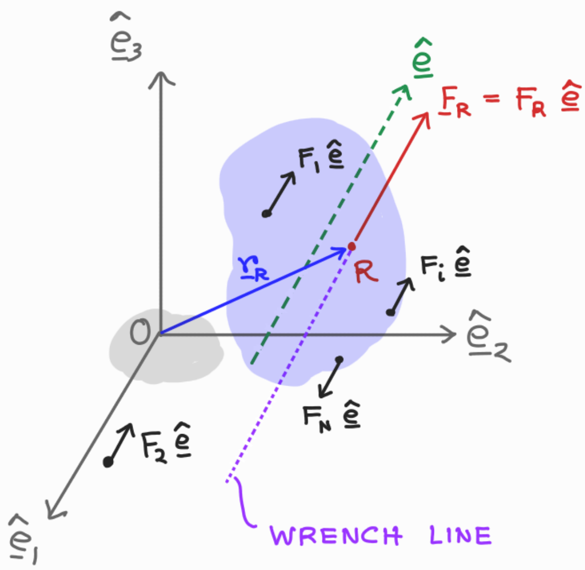
Center of Parallel Forces

{ Centre (UK Eng) vs Center (US Eng) }

Consider a force system having <u>ONLY</u> parallel forces, $\underline{F}_i = F_i\,\underline{\hat{e}}$ all acting along $\underline{\hat{e}}$ direction

The resultant force, $\underline{F}_R = \sum_{i=1}^{N} \underline{F}_i = \sum_{i=1}^{N} F_i\,\underline{\hat{e}} = F_R\,\underline{\hat{e}}$

Now, we want to locate point 'R', where the resultant wrench F_R acts, that is, find its position vector $\underline{r}_R$



For this, lets balance the moments about (say) 'O':

$\underline{M}_O$ = Net moment about 'O'

$$\Rightarrow \sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i = \underline{r}_R \times \underline{F}_R$$

$$\Rightarrow \sum_{i=1}^{N} \left(\underline{r}_i \times F_i\,\underline{\hat{e}}\right) = \underline{r}_R \times F_R\,\underline{\hat{e}}$$

$$\Rightarrow \left(\sum_{i=1}^{N} \underline{r}_i F_i\right) \times \underline{\hat{e}} = \left(\underline{r}_R F_R\right) \times \underline{\hat{e}}$$

$$\Rightarrow \left\{\left(\sum_{i=1}^{N} \underline{r}_i F_i\right) - \underline{r}_R F_R\right\} \times \underline{\hat{e}} = \underline{0}$$

← We can find $\underline{r}_R$ from this equation

$$\boxed{\underline{r}_R = \frac{\left(\sum_{i=1}^{N} \underline{r}_i F_i\right)}{F_R}}$$

Position vector of R

(when $F_R \neq 0$)

This equation is true for any arbitrary dir. $\underline{\hat{e}}$

$\underline{r}_R$ is also called the center of parallel forces, which is the action point of the resultant force $\underline{F}_R$

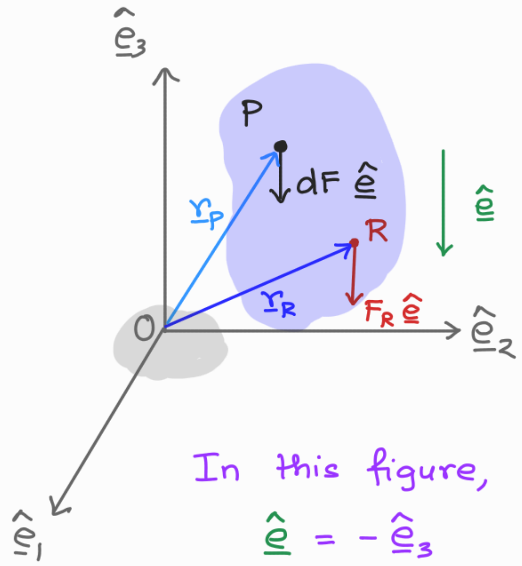
So far we have discussed force systems with concentrated or point forces. We can also have distributed force systems that are:

→ force distributed over an area, e.g. pressure or surface force

→ force distributed over a volume, e.g. gravitational force

In case of continuous distribution of parallel forces (e.g. gravity) we can consider an infinitesimal force dF acting at point P in the direction 'e',

The resultant force, $\underline{F}_R = \int dF \,\hat{\underline{e}}$



The position vector $\underline{r}_R$:

summation replaced with integration

$$\underline{r}_R = \frac{\int \underline{r}_P \, dF}{\int dF},$$

holds true for all dir $\hat{\underline{e}}$

In this figure, $\hat{\underline{e}} = -\hat{\underline{e}}_3$

Ex: If the force is due to gravity, then $dF = (dm)g$ and the center of parallel force system would coincide with the center of gravity

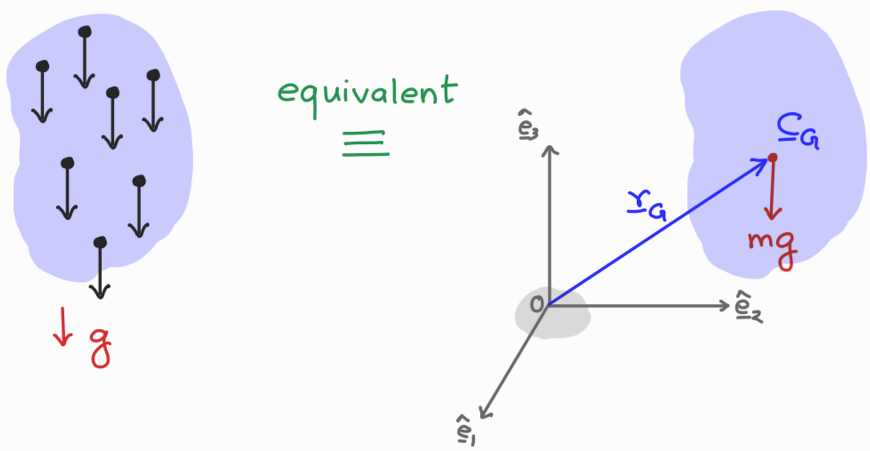
i.e. $\underline{r}_R = \underline{r}_G$ (Center of Gravity)

Center of Gravity, (COG) $\underline{r}_G = \dfrac{\int_{body} \underline{r}_p \, dm \, \cancel{g}}{\int_{body} dm \, \cancel{g}} = \dfrac{\int_{body} \underline{r}_p \, dm}{m} = \underline{r}_c$

total mass

Center of mass (COM)

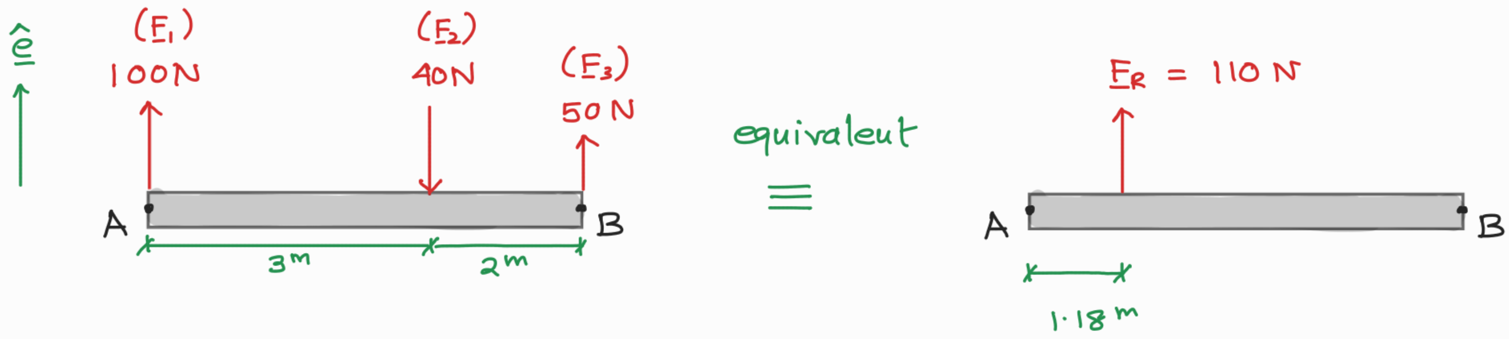
Therefore, COM and COG coincide when gravity is same through-out the body.



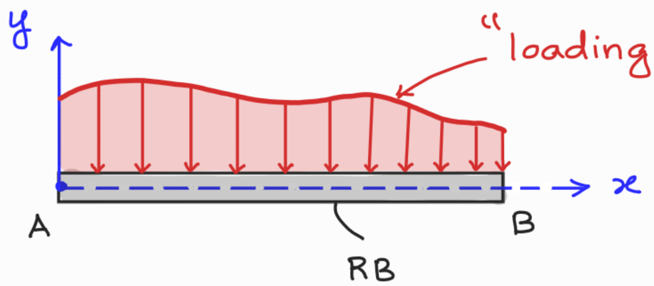
Coplanar & Parallel Distributed Force System

We have already seen equivalent force system for parallel and coplanar discrete force systems.

Example of an RB loaded with coplanar parallel discrete forces

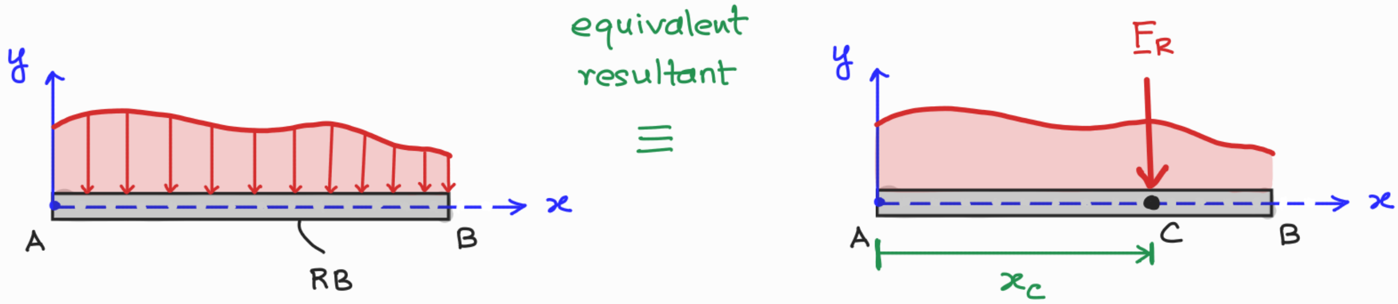


Now we will consider a co-planar and parallel force system with distributed forces



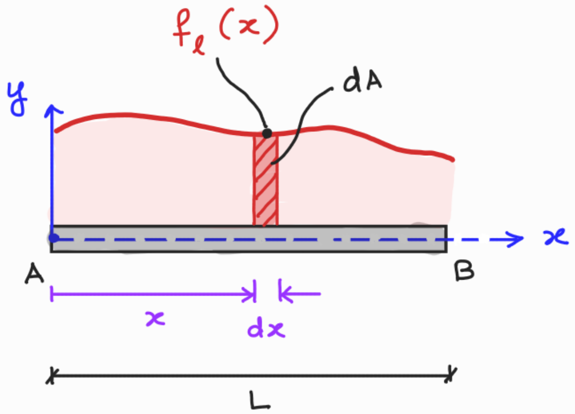
"loading curve" $f_\ell(x)$ representing the distribution of forces along an RB

Figure shows an RB under the influence of a line-distributed force system in the x-y plane.



$x_c \rightarrow$ x-coordinate of the centroid of the loading curve

We will use the two relations of equivalent force systems to establish equivalence



dA = area of the shaded region

$= f_\ell(x)\,dx$

The resultant force magnitude:

① F_R = Area under the loading curve

$$= \int dA = \int_{x=0}^{x=L} f_\ell(x)\,dx$$

② Using the second condition of force equivalence → moment about a point (say A) should be same for both systems

moment arm of dA

$$-\int_{x=0}^{x=L} \big(\overbrace{f_\ell(x)\,dx}^{dA}\big)\, x = -F_R\, x_R$$

negative sign is because the moment dir. is opposite to +z

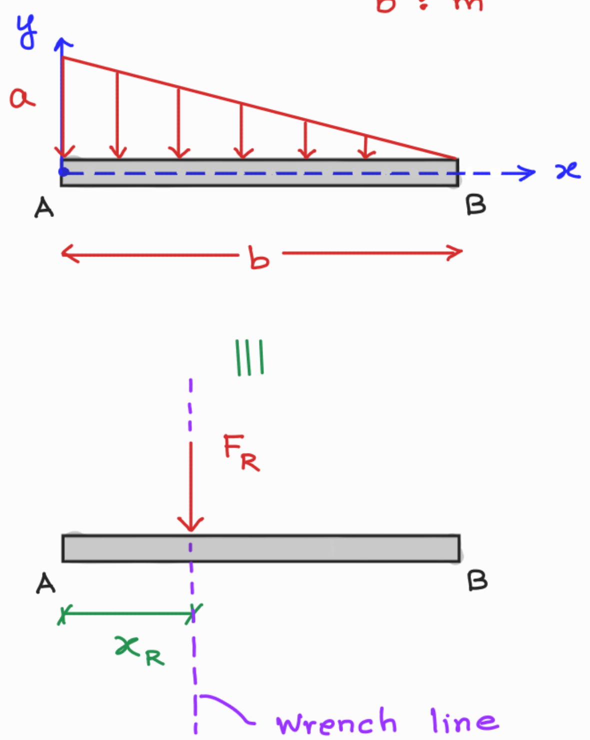
$$\Rightarrow \quad x_R = \frac{\int_{x=0}^{x=L} f_\ell(x)\, x\, dx}{F_R} = \frac{\int x\, dA}{\int dA} = x_c$$

> The simplest resultant F_R (and its WRENCH line) pass through the centroid in a coplanar and parallel distributed force system

Example 1:

a : N/m

b : m



Beam AB under the influence of a line distributed force system

$$f_\ell(x) = a\left[1 - \frac{x}{b}\right] \ (\text{N/m})$$

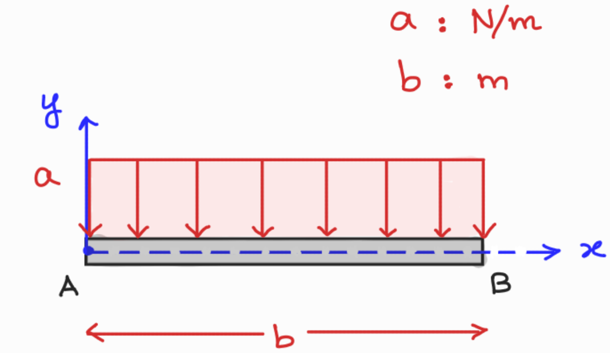
F_R = area under the loading curve

$= \frac{1}{2}\, a\, b \quad (\text{N})$

x_R = x-coordinate of centroid of the area under loading curve

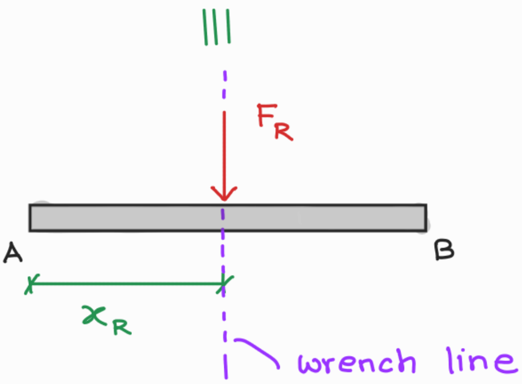
$= \frac{b}{3}$ (from A)

Example 2:



Beam AB under the influence of a line distributed force system

$$f_\ell(x) = a \ (N/m)$$

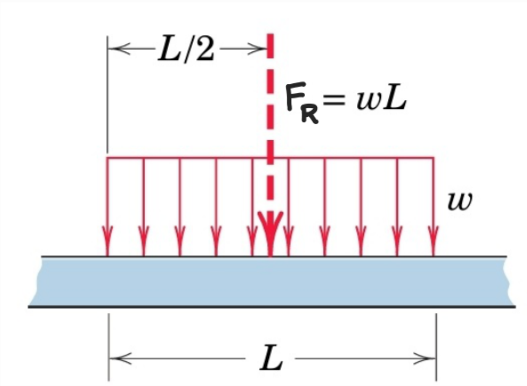


F_R = area under the loading curve

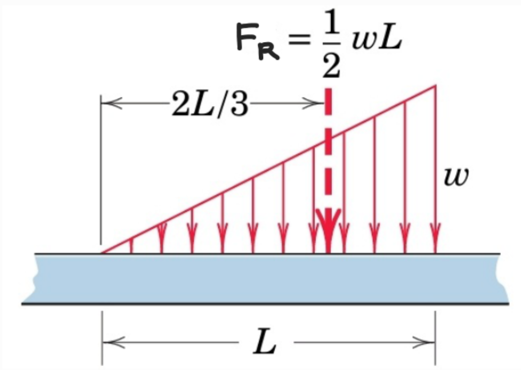
$= a\,b \quad (N)$

x_R = x-coordinate of centroid of the area under loading curve

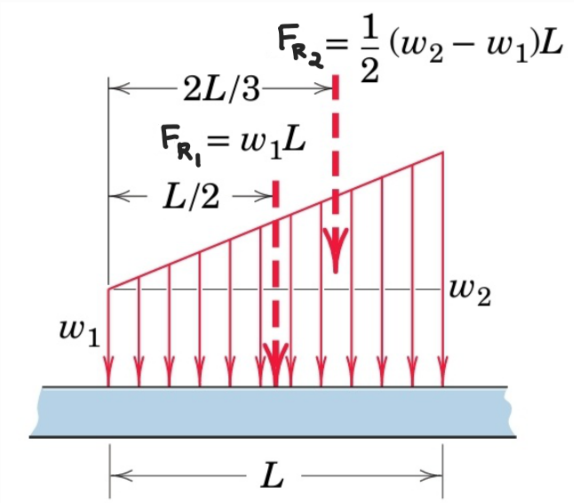
$= \underline{b} \quad$ (from A)



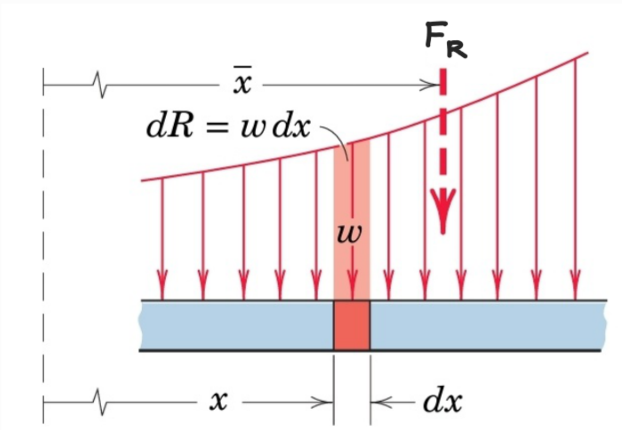
(a) Rectangular



(b) Triangular



(c) Trapezoidal



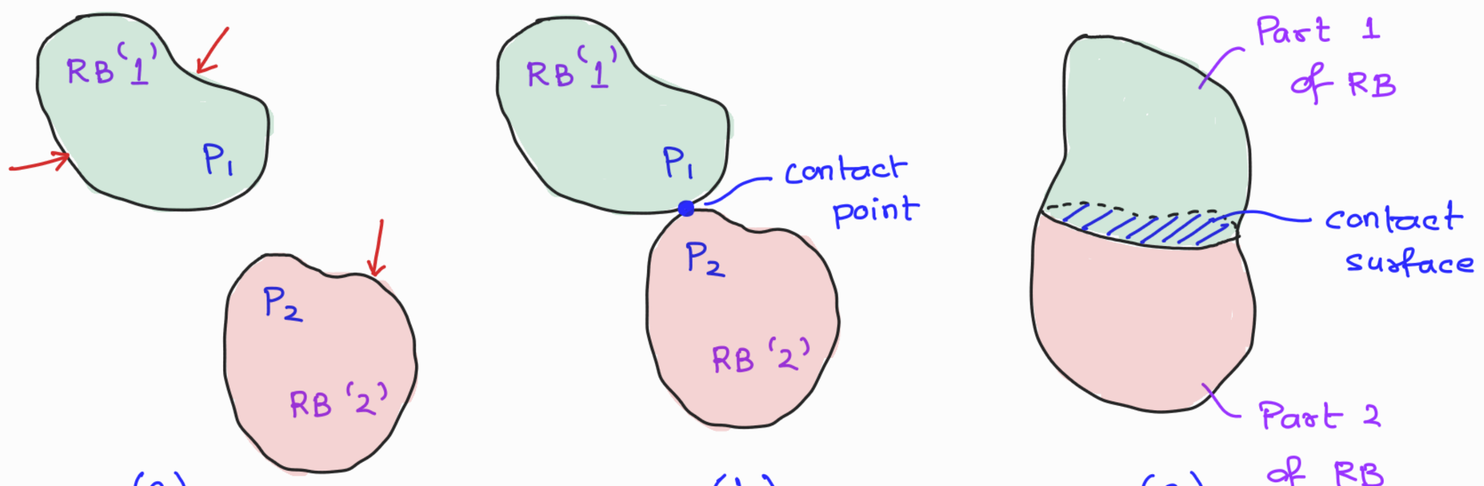
(d) Arbitrary shape

Interaction of forces between two RBs

I Newton's 3rd law

What can we say about loads of interaction between two bodies (or their parts) ?

Ex:



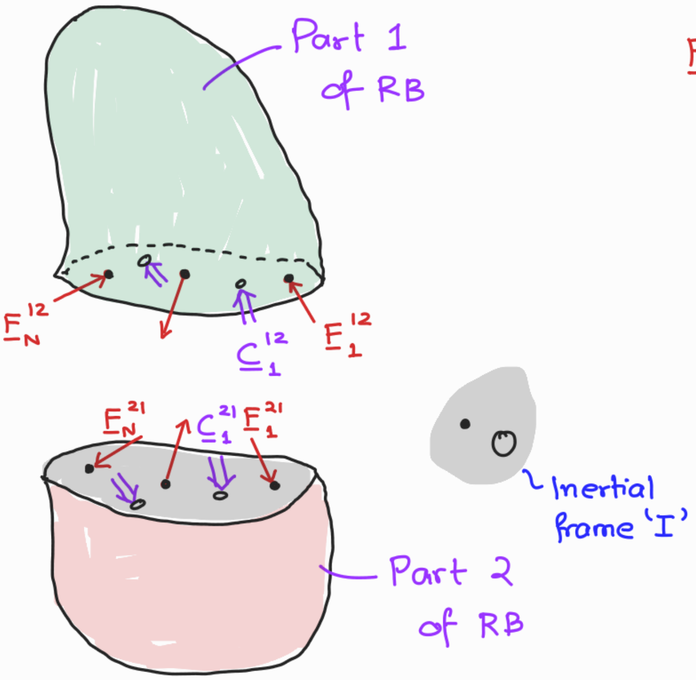
(a) Bodies at a distance

(b) Bodies in contact

(c) Two parts of the same body

" Loads (forces and moments) of interaction between two bodies (or their parts) have equal magnitudes and opposite directions. "

meaning if two bodies RB '1' and RB '2' (or their parts) interact at a contact point or over a contact surface or a distance, then the force sum and moment sum abt a pt O (a fixed pt of an inertial frame) for the two RBs are related by Newton's 3rd law of motion.



$\underline{F}^{12} \leftarrow$ force exerted by part '2' of RB on the part '1' of RB

$\underline{F}^{21} \leftarrow$ force exerted by part '1' of RB on the part '2' of RB

$\underline{C}^{12}$, $\underline{C}^{21}$ are couples defined similarly

Then, Newton's 3rd law of motion states that :

Force sums are equal & opposite :

$$\sum_{i=1}^{N} \underline{F}_i^{21} = -\sum_{i=1}^{N} \underline{F}_i^{12} \Rightarrow \boxed{\underline{F}_R^{21} = -\underline{F}_R^{12}}$$

Moment sums abt O (a pt in inertial frame) are equal and opposite :

$$\sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i^{12} + \sum_{j=1}^{M} \underline{C}_j^{12} = -\left(\sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i^{21} + \sum_{j=1}^{M} \underline{C}_j^{21}\right)$$

$$\Downarrow$$

$$\boxed{\underline{M}_O^{21} = -\underline{M}_O^{12}}$$

Proof of Newton's 3rd law can be derived using Euler's two axioms.

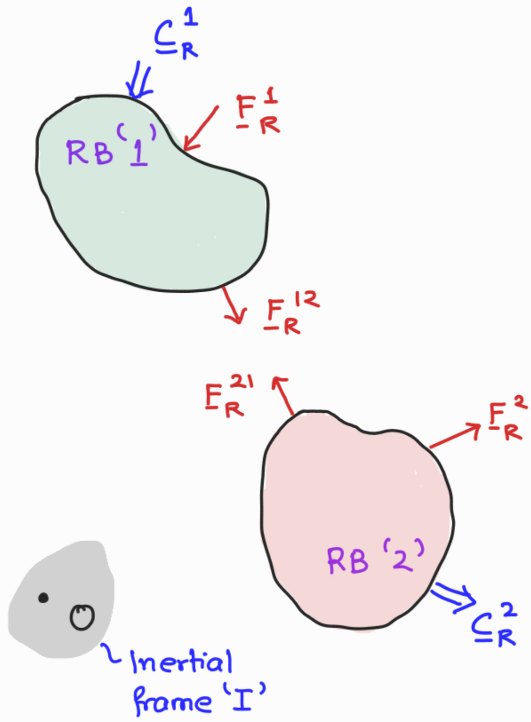
Forces and moments acting on RB 1

$\underline{F}_R^1$, $\underline{C}_R^1$ = Net ext. force and couple

$\underline{F}_R^{12}$ = Ext. force applied by RB 2

$\underline{M}_O^1$ = Net ext. moment abt O due to $\underline{F}_R^1$ & $\underline{C}_R^1$

$\underline{M}_O^{12}$ = Ext. moment abt O due to $\underline{F}_R^{12}$



Forces and moments acting on RB 2

$\underline{F}_R^2$, $\underline{C}_R^2$ = Net ext. force and couple

$\underline{F}_R^{21}$ = Ext. force applied by RB 1

$\underline{M}_O^2$ = Net ext. moment abt O due to $\underline{F}_R^2$ and $\underline{C}_R^2$

$\underline{M}_O^{21}$ = Ext. moment abt O due to $\underline{F}_R^{21}$

Using Euler's two axioms (valid for inertial frame), we can write:

For RB 1

$$\dot{\underline{P}}_{1|I} = \underline{F}_R^1 + \underline{F}_R^{12}$$

$$\dot{\underline{H}}_{O1|I} = \underline{M}_O^1 + \underline{M}_O^{12}$$

For RB 2

$$\dot{\underline{P}}_{2|I} = \underline{F}_R^2 + \underline{F}_R^{21}$$

$$\dot{\underline{H}}_{O2|I} = \underline{M}_O^2 + \underline{M}_O^{21}$$

For RB1 + RB2 combined together and treated as one system

union

$$\dot{\underline{P}}_{1\cup 2|I} = \underline{F}_R^1 + \underline{F}_R^2 = \dot{\underline{P}}_{1|I} + \dot{\underline{P}}_{2|I} = \underline{F}_R^1 + \underline{F}_R^{12} + \underline{F}_R^2 + \underline{F}_R^{21}$$

$$\Rightarrow \underline{F}_R^{12} = -\underline{F}_R^{21}$$

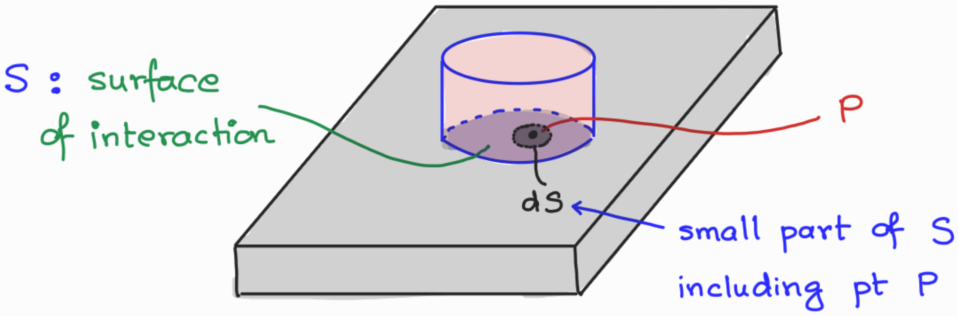
$$\dot{\underline{H}}_{O1\cup O2|I} = \underline{M}_O^1 + \underline{M}_O^2 = \dot{\underline{H}}_{O1|I} + \dot{\underline{H}}_{O2|I} = \underline{M}_O^1 + \underline{M}_O^{12} + \underline{M}_O^2 + \underline{M}_O^{21}$$

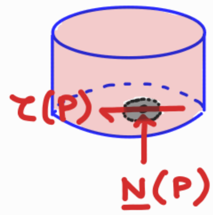
$$\Rightarrow \underline{M}_O^{12} = -\underline{M}_O^{21}$$

II Coulomb's axioms of friction for two RBs in contact

The mechanical interaction at the surfaces of contact of two RBs can be modeled by a distributed surface force whose components are normal and tangent to the contact surfaces.

Normal reaction — Friction force

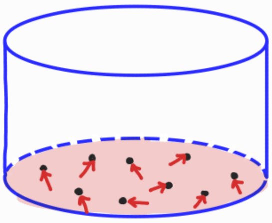




At an interface point P, the resultant contact for exerted by one body upon the other may be separated into component forces :

① normal force $\underline{N}(P)$ ← mutual pushing (or compression) of one body by the other perpendicular to the interface
(normal component)

② friction force $\underline{\tau}(P)$ ← the mutual resistance to sliding of one body surface over the other
(tangential component)

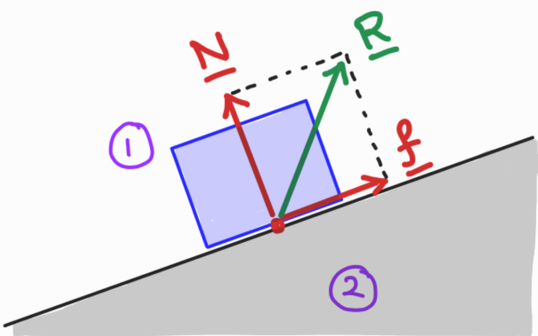


the distribution of contact force over S is usually unknown

However, the information about the contact force is required before any problem involving friction can be solved. (not easy)

To overcome this hurdle, we consider the equivalent normal & tangential force distributions and find a resultant normal force $\underline{N}$ and a resultant friction force $\underline{f}$:

Contact forces on RB 1 by RB 2



Resultant normal force over S , $\underline{N} = \int_S \underline{n}(P)\, dS$

Resultant friction force over S , $\underline{f} = \int_S \underline{\tau}(P)\, dS$

and

Resultant contact force , $\underline{R} = \underline{N} + \underline{f}$

1> Axiom of static friction : If 'no-slip' condition is valid for two RBs in contact, then the magnitude of frictional force between the surface of interaction is :

$$|\underline{f}| < \mu_s |\underline{N}|$$

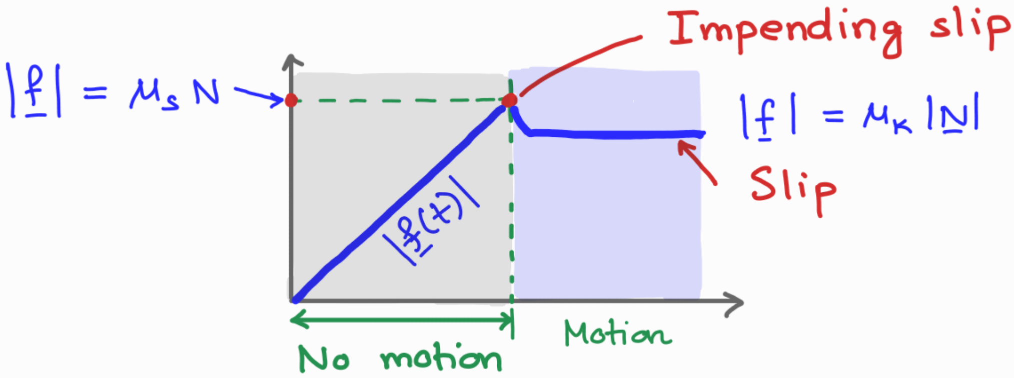
magnitude of resultant frictional force

coeff. of static friction

magnitude of resultant normal force

If "impending slip" condition is valid for the RBs, then maximum frictional force is encountered:

$$|\underline{f}| = \mu_s |\underline{N}|$$



$$\boxed{\mu_k \leq \mu_s}$$

2) Axiom of dynamic friction: If two RBs in contact are slipping / sliding relative to each other, then the magnitude of the frictional force is:

coeff. of kinetic friction

$$|\underline{f}| = \mu_k |\underline{N}|$$

- μ_s, μ_k are independent of the relative sliding velocities, area of contact between RBs
- μ_s, μ_k depend only on the two surfaces (material & finish)

FREE BODY DIAGRAM (FBDs)

For applying Euler's axioms to an RB (or a system of RBs), it is necessary that

a) the RB (or the system) be sketched in isolation from its surroundings, and

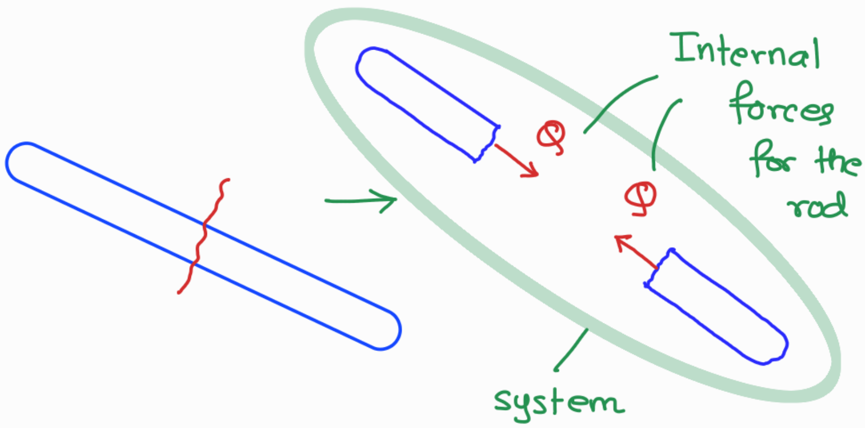
b) all external forces and moments exerted by the surroundings on the RB (or the system) be drawn on it

Such a diagram is called FREE BODY DIAGRAM (FBD) and is the most important single step in the solution of problems in mechanics

Before drawing the FBD, recall

ⓐ Forces/Couples → Contact (applied by direct physical contact)
→ Body (applied remotely, e.g. gravity)

ⓑ Forces/Couples → External (applied on a system)
→ Internal



(c) Application of force on a system may be accompanied by reaction force system (both forces and/or moments), and both applied and reaction forces may be either point or distributed. (Will look into force systems caused by supports in the next lecture)

Construction of Free-Body Diagrams

Step 1: Decide which system to isolate. The system chosen should usually involve one or more of the desired unknown quantities (such as reactions)

Step 2: Isolate the chosen system by drawing a diagram which represents its complete external boundary.

The boundary defines the isolation of the system from ALL other attracting or contacting bodies, which are considered removed. (Most crucial step!)

Step 3: Identify all forces which act on the isolated sys. as applied by the removed contacting and attracting bodies, and represent them in their proper positions on the diagram of the isolated system.

(a) Make sure you have identified all contact forces on the entire boundary

(b) Include all body forces, such as weight due to gravity, where appreciable

(c) Represent all known forces and moments with their magnitude and direction

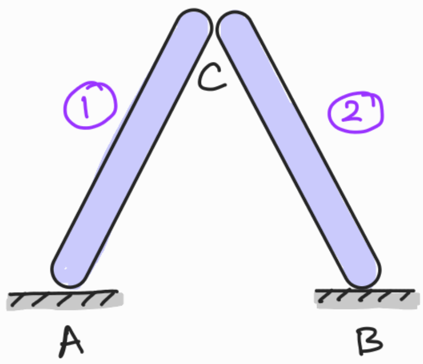
(d) Each unknown force/moment should be represented by a vector with unknown magnitude and/or direction

Arbitrarily assign a direction if it is unknown

Step 4: Choose a coordinate system and show the coord. axes on the diagram.

When working in an inertial frame of reference, inertial forces like centrifugal forces, Coriolis force, $m\underline{a}$, etc. are not shown in an FBD!

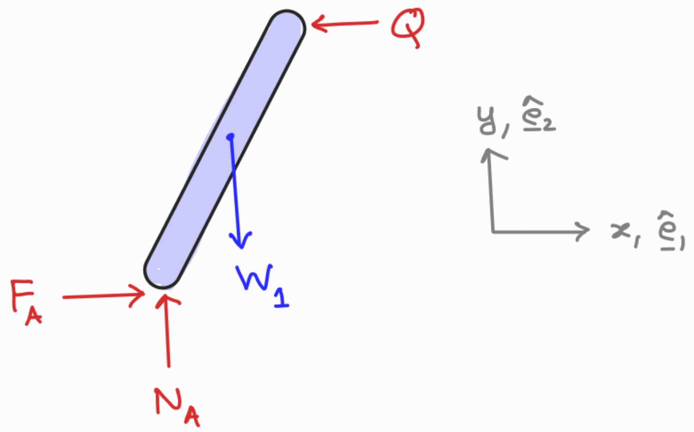
Example: Consider two rods of certain weight touching each other and stranded on rough surfaces.



- Draw FBD for rod 1
- Draw FBD for rod 2
- Draw FBD for rods 1 & 2 considered as a system

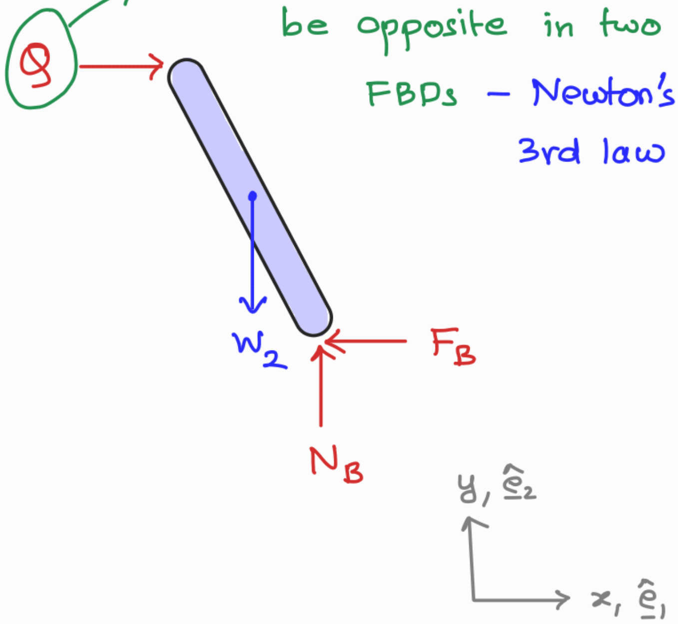
FBD of rod 1

- Isolate rod 1
- Identify contact forces
- Identify body forces (equivalent body force acting at COM)



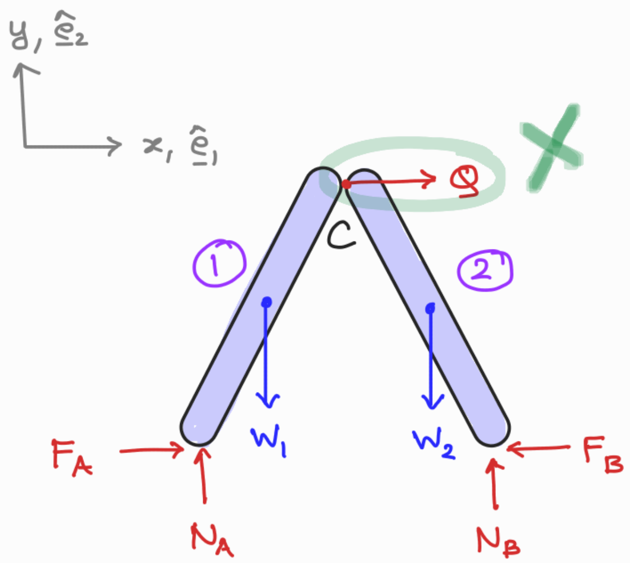
FBD of rod 2

- Isolate rod 2
- Identify contact forces
- Identify body forces (equivalent body force acting at COM)



Direction of Q should be opposite in two FBDs – Newton's 3rd law

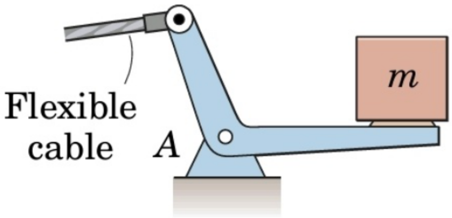
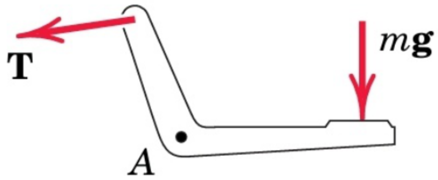
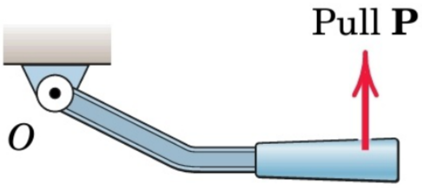
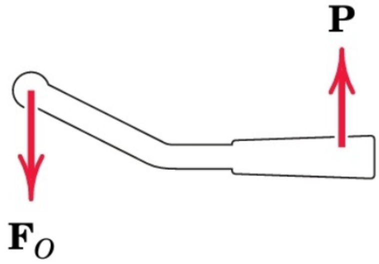
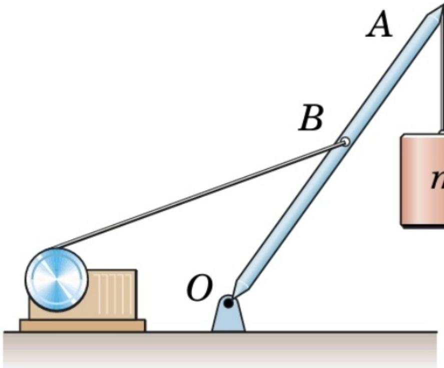
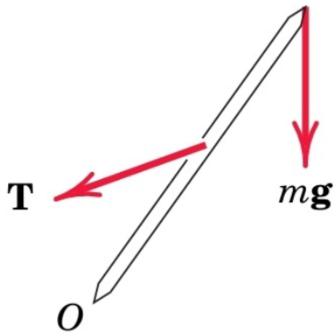
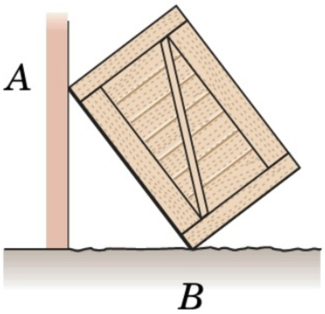
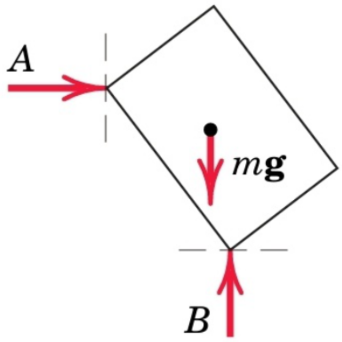
FBD of two rods treated as a system



- Isolate rods ① and ②
- Identify contact forces
- Identify body forces (equivalent body force acting at COM)

- Q must not appear on FBD if both rods are considered as one system (as Q would be an internal (not external) contact force)

Can you guess the correct FBDs?

	Body	Incomplete FBD
1. Bell crank supporting mass m with pin support at A.		
2. Control lever applying torque to shaft at O.		
3. Boom OA, of negligible mass compared with mass m. Boom hinged at O and supported by hoisting cable at B.		
4. Uniform crate of mass m leaning against smooth vertical wall and supported on a rough horizontal surface.		
5. Loaded bracket supported by pin connection at A and fixed pin in smooth slot at B.