Recap

In the last lecture, we introduced the two Euler's axioms for relating the translational and rotational motion of an RB to the causal forces and moments, respectively.

There exists an inertial frame I such that Euler's axioms hold true which leads to the governing eqn of motion of an RB:

1. $\dot{\underline{P}}\Big|_I = \underline{F}_{net}^{ext} = m\,\underline{v}_{C|I}$ (Recall $\underline{P}_{|F} = m\,\underline{v}_{P|F}$)

2. $\dot{\underline{H}}_{O|I} = \underline{M}_{O,net}^{ext}$

→ 6 scalar equations

⇓

(a) Necessary and sufficient for solving motion of an RB

$\Rightarrow$ (3 translation + 3 rotation $\equiv$ 6 degrees of freedom)

(b) NOT sufficient to solve for the motion of a deformable body

For applying Euler's axioms to an RB (or a system of RBs), it is necessary that

a) the RB (or the system) be sketched in isolation from its surroundings, and

b) the external forces and moments exerted by the surroundings on the RB (or the system) be drawn on it

Such a diagram is called FREE BODY DIAGRAM (FBD)

Note that the forces exerted by one part of the RB (or system) on another part of the RB (or system) are called internal forces

These internal forces should not be shown in an FBD since they do not appear in Euler's axioms.

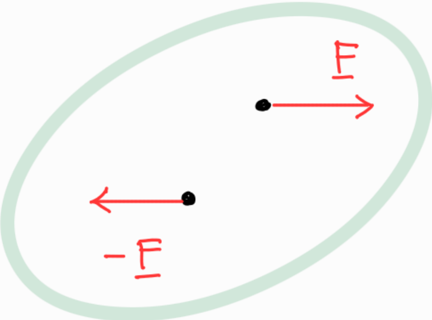
To apply Euler's axioms to a system, we need to compute forces ($\underline{F}$) and moments ($\underline{M}_o$) due to the external loads acting on it.

We shall call a load system consisting of forces and moments simply as a force system

In this lecture, we will learn how to replace a given system of forces by a simpler equivalent system.

An important concept associated with the effect of a force on on RB is the moment of a force about a point and abt an axis. We will also recap couple caused by two parallel forces with same magnitude but opposite direction. As you will see that we can replace any system of forces acting on an RB by an equivalent system consisting of one force acting on an RB and one couple!

Recap of Couple :



Two equal and opposite forces acting along different (parallel) lines of action

→ Couple can have only a turning (rotating) effect

→ Couple stays the same about any point in space

$\Downarrow$

FREE VECTOR ! $\Leftrightarrow$ No change on shifting the location!

→ Representation of couple:

Directed double lined arrow

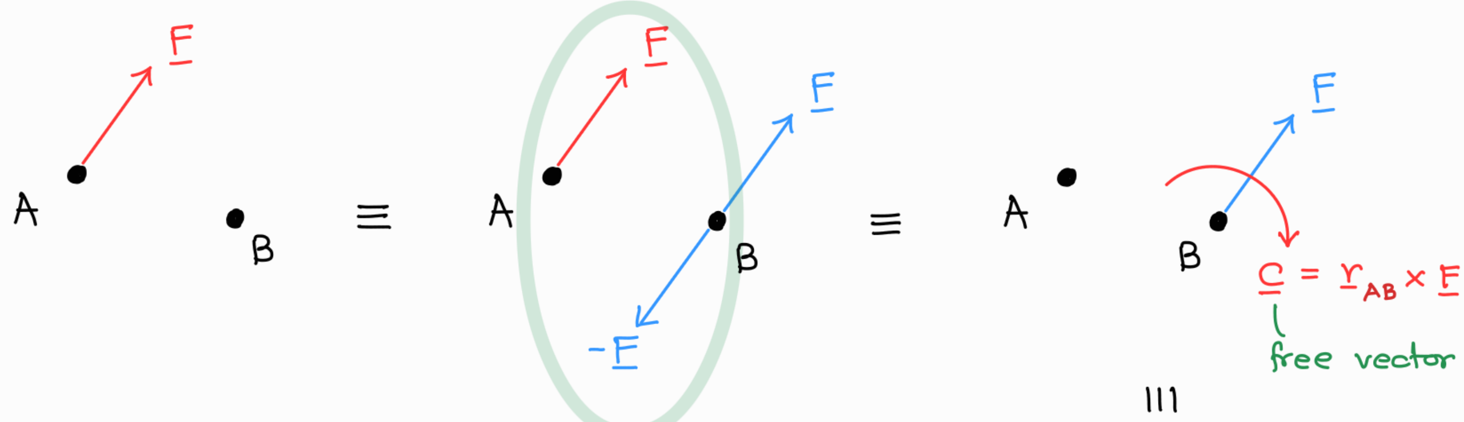
length : Magnitude of $\underline{C}$

direction : Direction of $\underline{C}$

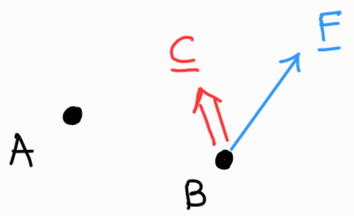
Curled arrow

Direction of $\underline{C}$ given by right-hand-thumb rule

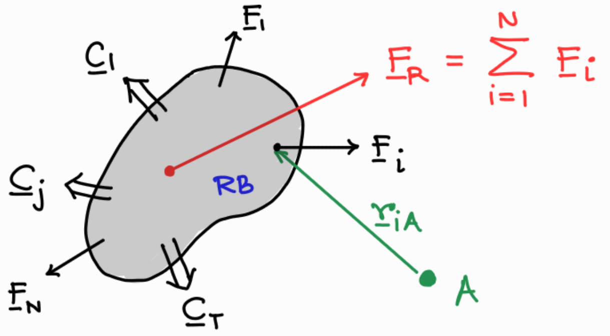
What happens when $\underline{F}$ is shifted from a pt A to another pt B ? It generates a couple as well!



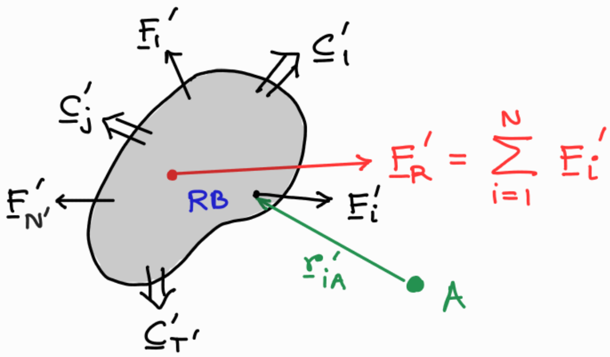
$$\text{Force} \xrightarrow[\text{to another pt}]{\text{when shifted}} \begin{bmatrix}\text{Same}\\ \text{Force}\end{bmatrix} + \begin{bmatrix}\text{One}\\ \text{couple}\end{bmatrix}$$



Equivalent Force Systems



Force System 1



Force System 2

$\underline{F}_i$ and $\underline{C}_j$ are part of Force system 1

$\underline{F}_i'$ and $\underline{C}_j'$ are part of Force system 2

Note, in general $\begin{cases} \underline{F}_i \neq \underline{F}_i' \\ \underline{C}_j \neq \underline{C}_j' \\ \underline{r}_{iA} \neq \underline{r}_{iA}' \end{cases}$

(N need not be equal to N')

(T need not be equal to T')

The force systems 1 and 2 are said to be equivalent if both the force systems result in the same net force and same net moment about a point 'A' $\longrightarrow$ could be anywhere in space

$$\underline{F}_R = \underline{F}_R' \Rightarrow \sum_{i=1}^{N} \underline{F}_i = \sum_{i=1}^{N'} \underline{F}_i' \quad —\ (a)$$

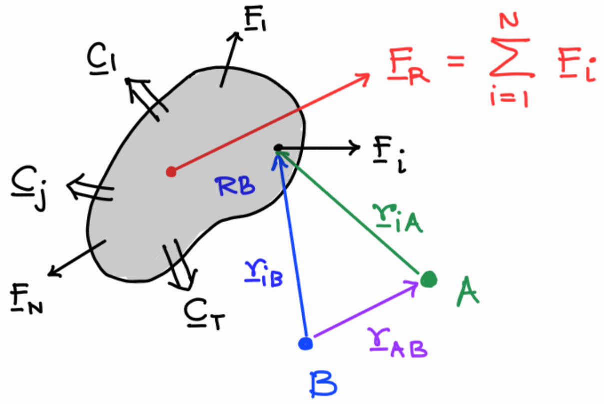
$\Leftrightarrow$ AND

$$\underline{M}_A = \underline{M}_A' \Rightarrow \underbrace{\sum_{i=1}^{N} \underline{r}_{iA} \times \underline{F}_i + \sum_{j=1}^{T} \underline{C}_j}_{\underline{M}_A} = \underbrace{\sum_{i=1}^{N'} \underline{r}_{iA}' \times \underline{F}_i' + \sum_{j=1}^{T'} \underline{C}_j'}_{\underline{M}_A'} \quad —\ (b)$$

Equivalent force systems result in same motion of an RB.

If conditions (a) and (b) have been found true for one point A in space, it follows that (b) will hold true for every other point (say 'B')

Moment abt pt B due to force system 1



$$\underline{M}_B = \sum_{i=1}^{N} \underline{r}_{iB} \times \underline{F}_i + \sum_{j=1}^{T} \underline{C}_j$$

$$= \sum_{i=1}^{N} (\underline{r}_{AB} + \underline{r}_{iA}) \times \underline{F}_i + \sum_{j=1}^{T} \underline{C}_j$$

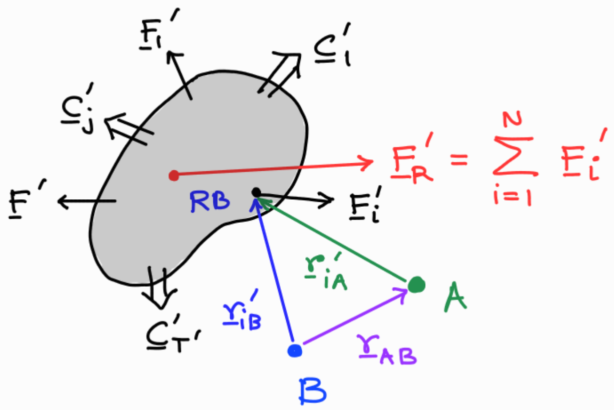
$$= \underbrace{\sum_{i=1}^{N} \underline{r}_{iA} \times \underline{F}_i + \sum_{j=1}^{T} \underline{C}_j}_{\underline{M}_A} + \sum_{i=1}^{N} \underline{r}_{AB} \times \underline{F}_i$$

same for all i

$$= \underline{M}_A + \underline{r}_{AB} \times \underbrace{\left(\sum_{i=1}^{N} \underline{F}_i\right)}_{\underline{F}_R} \Rightarrow \underline{M}_B = \underline{M}_A + \underline{r}_{AB} \times \underline{F}_R \quad —\ (*)$$

Similarly, for force system 2

$$\underline{M}_B' = \underline{M}_A' + (\underline{r}_{AB} \times \underline{F}_R')$$



Since $\underline{M}_A = \underline{M}_A'$ and $\underline{F}_R = \underline{F}_R'$

$\Downarrow$

Therefore, the moment sum of two equivalent force systems is the same about <u>ANY</u> point

Thus if $\underline{F}_R = \underline{F}_R'$ <u>AND</u> $\underline{M}_A = \underline{M}_A'$,

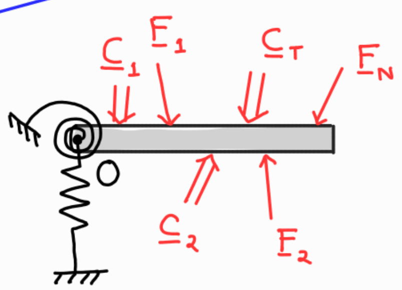
$$\Rightarrow \underline{M}_B = \underline{M}_B' \quad \text{or} \quad \underline{M}_C = \underline{M}_C', \text{ etc.}$$

where A, B, C are any points in space.

Note: $\underline{M}_A \neq \underline{M}_B \neq \underline{M}_C$ in general

In addition to application of Euler's axioms, the motivation for studying equivalent force systems comes from several practical applications

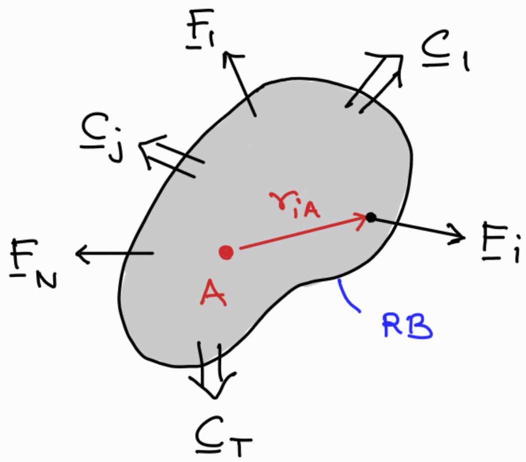
<u>Example</u>



For design purposes, we may need to find the resultant effect of all the forces & couples at the support O to calculate the spring stiffnesses for resisting the forces and moments.

This is one of the motivations to study equivalent force systems.

Now, since any two equivalent force system will result into identical motion of an RB, we can replace a given force system (N forces, T couples) by an equivalent system with just one force ($\underline{F}_R$) acting at a point A on RB along with a couple $(\underline{C}_R)_A$



Forces : $\underline{F}_i$ $\quad i = 1, 2, \ldots, N$
Couples : $\underline{C}_j$ $\quad j = 1, 2, \ldots, T$

Original forces & couples acting on RB

'A' → point where we want to shift the forces and couples

① Couple ⇒ free vector ⇒ No change on shifting
↳ same at any point in space

② Force $\xrightarrow{\text{on shifting}}$ [Same force] + [a couple]

What is the RESULTANT equivalent force & couple at A ?

Resultant force system at A

Net force ($\underline{F}_R$) $= \sum_{i=1}^{N} \underline{F}_i$

Net couple ($\underline{C}_R$) $= \sum_{j=1}^{T} \underline{C}_j + \sum_{i=1}^{N} \underline{r}_{iA} \times \underline{F}_i$

Original couples

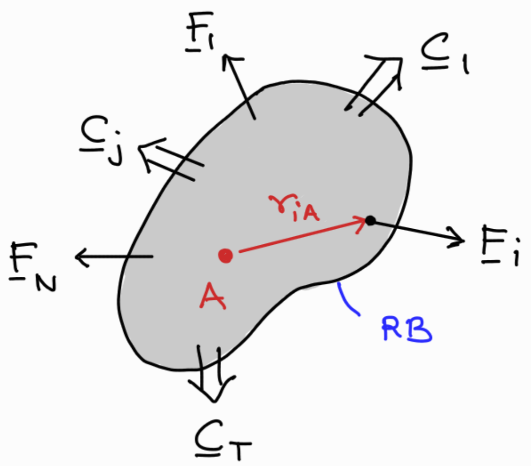
Couple contribution due to shifting of force $\underline{F}_i$ from pt 'i' to point 'A'

(subscript R is for resultant)

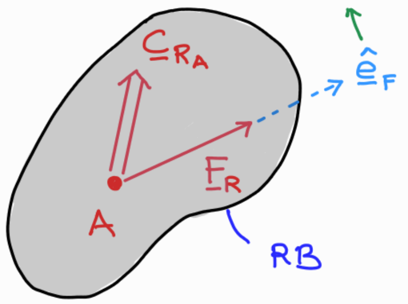
⇒ Any force system can be reduced to a single resultant force $\underline{F}_R$ and a single resultant couple $\underline{C}_{R_A}$ at point A

Define $\hat{\underline{e}}_F = \dfrac{\underline{F}_R}{|\underline{F}_R|}$

unit vector along $\underline{F}_R$



≡



Furthermore,

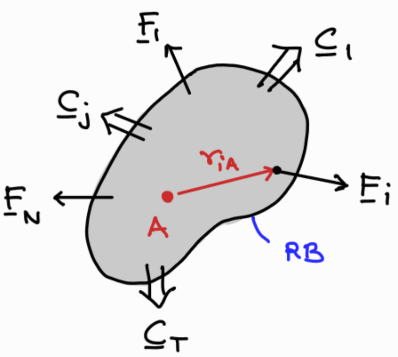
$\underline{C}_{R_A}$ can be resolved into components parallel to $\underline{F}_R$ and perpendicular to $\underline{F}_R$

$$\Rightarrow \quad \underline{C}_{R_A} = \underline{C}_{R_A}^{\parallel} + \underline{C}_{R_A}^{\perp}$$

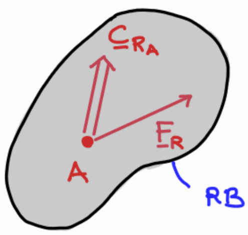
$$\underline{C}_{R_A}^{\parallel} = (\underline{C}_{R_A} \cdot \hat{\underline{e}}_F)\,\hat{\underline{e}}_F \quad \text{and} \quad \underline{C}_{R_A}^{\perp} = \underline{C}_{R_A} - \underline{C}_{R_A}^{\parallel}$$

Now, what is the couple at a different point B?

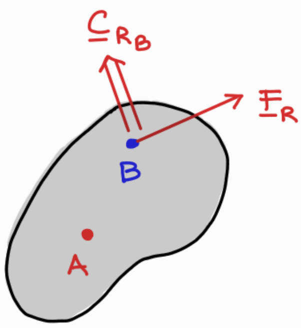
Recall: $\quad \underline{C}_{R_B} = \underline{C}_{R_A} + (\underline{r}_{AB} \times \underline{F}_R)$



≡



≡



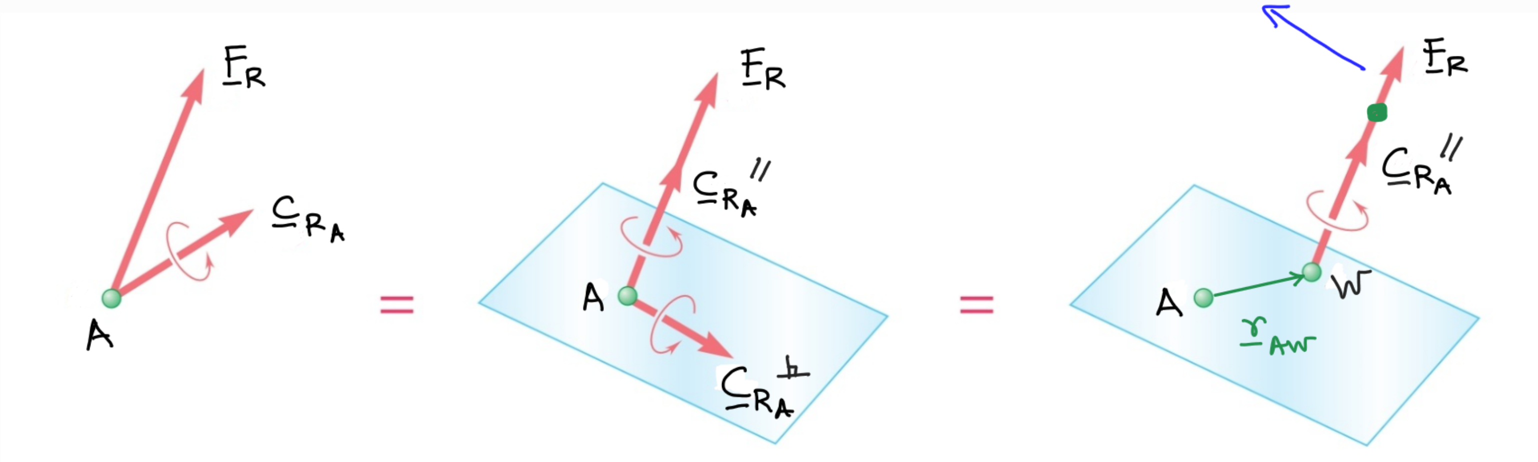
$$\underline{C}_{R_B} = \underline{C}_{R_A} + (\underline{r}_{AB} \times \underline{F}_R)$$

$$= \underline{C}_{R_A}^{\parallel} + \underbrace{\underline{C}_{R_A}^{\perp} + \underline{r}_{AB} \times \underline{F}_R}$$

A point 'W' can be chosen s.t.

Constraint $\rightarrow \underline{C}_{R_A}^{\perp} + \underline{r}_{AW} \times \underline{F}_R = \underline{0}$

The equivalent force system at point W is called "WRENCH"



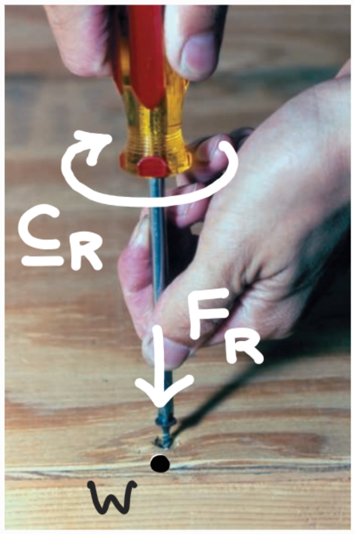
NOTE: The choice of pt W is not arbitrary. The pt. W must satisfy

$$(\underline{F}_R)_A = (\underline{F}_R)_W \quad \& \quad \underline{C}_{R_A}^{\perp} + \underline{r}_{AW} \times \underline{F}_R = \underline{0}$$

WRENCH

A force system where the net resultant of force and moment have the same direction!

e.g. force system of a screw driver



For a general 3D force system, simplest resultant is a WRENCH

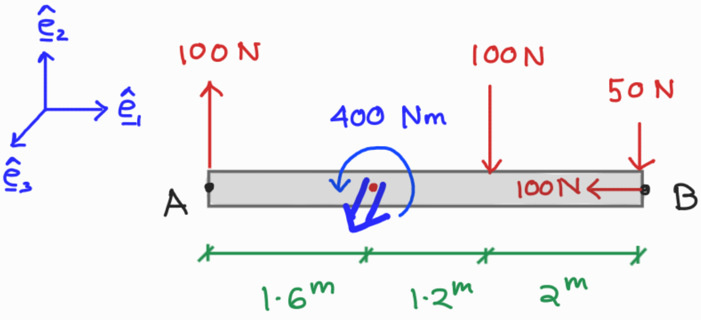
For some special cases the simplest resultant force system can simply be just one force ($\underline{F}_R$) placed at a special point OR a single couple $\underline{C}_{R_A}$ only (if $\underline{F}_R = \underline{0}$)

(1) Co-planar force systems
(2) Parallel force systems

$\Big\}$ special cases

(1) Coplanar Force Systems

All forces are in one plane, and couples (if any) are normal to the plane of action of forces



Suppose

(a) all forces are in $\hat{e}_1 - \hat{e}_2$ plane

(b) all couples are along $\hat{e}_3$ only

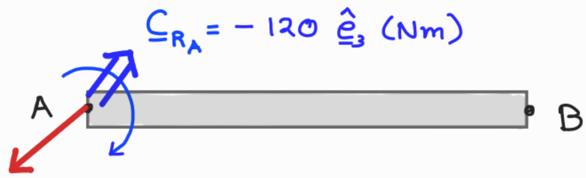
$$\underline{C}_{R_A} = C_{R_{Az}} \hat{e}_3$$

$$\Rightarrow \underbrace{\underline{C}_{R_A}^{\,//} = \underline{0}}$$

$\Downarrow$

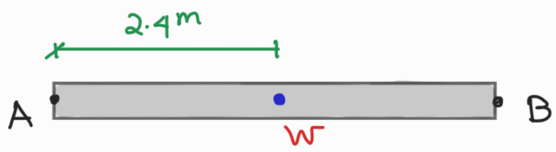
Simplest resultant (wrench) has $\underline{F}_R$ only (just one force)

equivalent |||

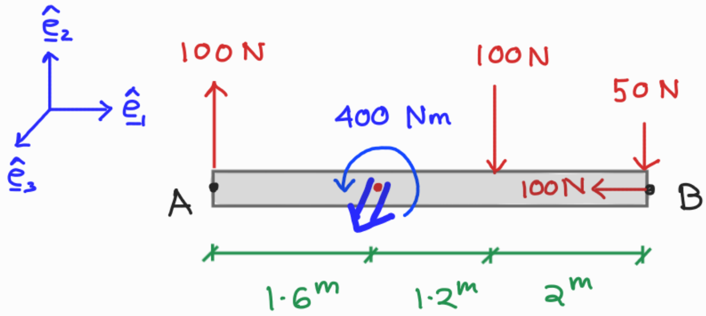


$\underline{F}_R = -100\ \hat{e}_1 + 50\ \hat{e}_2$ (N)

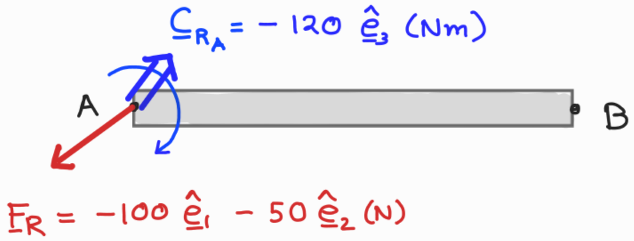
equivalent |||
WRENCH



$\underline{F}_R = -100\ \hat{e}_1 - 50\ \hat{e}_2$ (N)

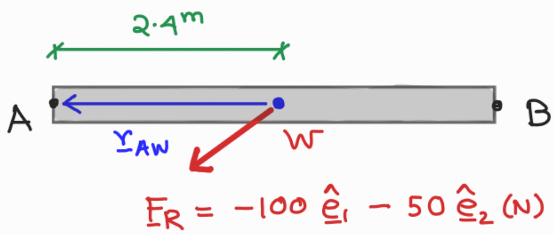


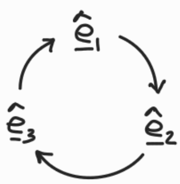
equivalent |||



equivalent |||
WRENCH

Resultant force
sys. at W





Calculation steps

$$F_R = -100\,\hat{e}_1 + (100-100-50)\,\hat{e}_2$$
$$= -100\,\hat{e}_1 - 50\,\hat{e}_2 \quad (N)$$

C_{R_A} = Resultant couple at pt A

$$= \Big[(100)\cdot(0) + (-100)\cdot(1.6+1.2) + (-50)\cdot(1.6+1.2+2) + 400 \Big]\,\hat{e}_3 + (-100)\cdot(0)\,\hat{e}_3$$

$$= -120\,\hat{e}_3 \quad (Nm)$$

Note: $C_{R_B} \neq C_{R_A}$

Wrench location:

$$r_{AW} \times F_R + C_{R_A}^{\perp} = 0$$

because A is the origin

$$\Rightarrow (-r_1\,\hat{e}_1 - r_2\,\hat{e}_2) \times (-100\,\hat{e}_1 - 50\,\hat{e}_2) - 120\,\hat{e}_3 = 0$$

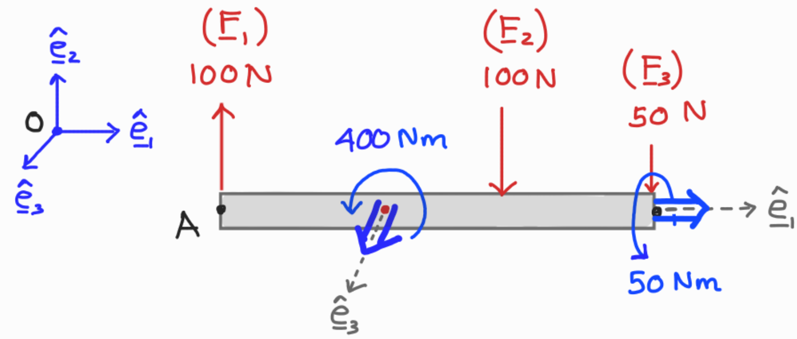
$$\Rightarrow \underbrace{50\,r_1 - 100\,r_2 = 120}_{\text{Equation of wrench line}}$$

If $r_2 = 0$, then $r_1 = \frac{120}{50} = 2.4^{m}$

(2) Parallel Force Systems

It comprises of forces which are all acting parallel to <u>one line</u> and couples (if any) have components only perpendicular to that line

Ex: all forces are (say) parallel to $\hat{\underline{e}}_2$ line, and couples present are along $\hat{\underline{e}}_1$ and/or $\hat{\underline{e}}_3$

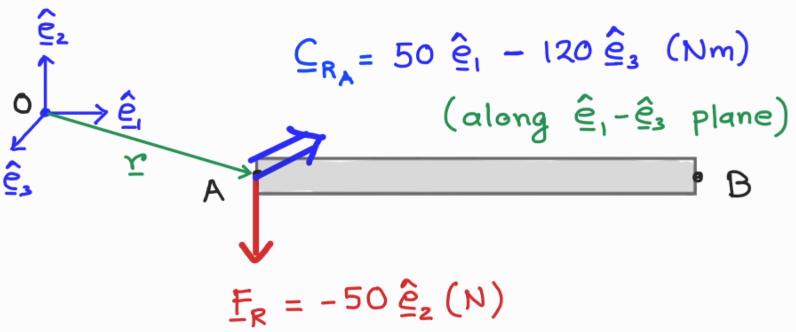


magnitude　　direction

$$\underline{F}_i = F_{iy}\,\hat{\underline{e}}_2$$

$$\underline{C}_i = C_{ix}\,\hat{\underline{e}}_1 + C_{iz}\,\hat{\underline{e}}_3$$

equivalent |||



$$\underline{F}_R = F_{Ry}\,\hat{\underline{e}}_2$$

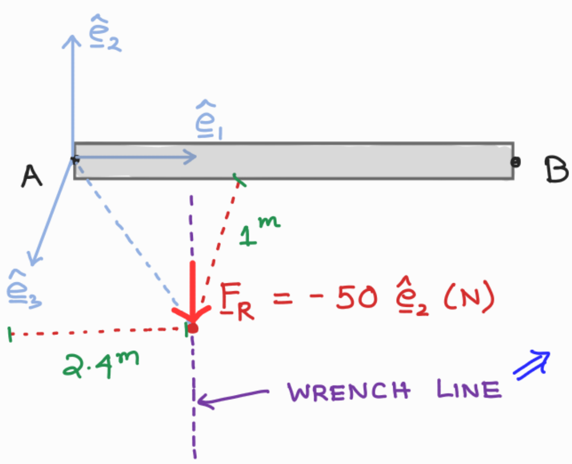
and

$$\underline{C}_{R_A} = C^x_{R_A}\,\hat{\underline{e}}_1 + C^z_{R_A}\,\hat{\underline{e}}_3$$

Since $\underline{r} \times \underline{F}_R$ is $\perp$ to $\underline{F}_R$

$$\Rightarrow \underline{C}_{R_A}^{\,/\!/} = \underline{0}$$

equivalent |||
WRENCH



Application of resultant force $\underline{F}_R$ anywhere on this line would lead to a <u>WRENCH</u> system