Chapter 2

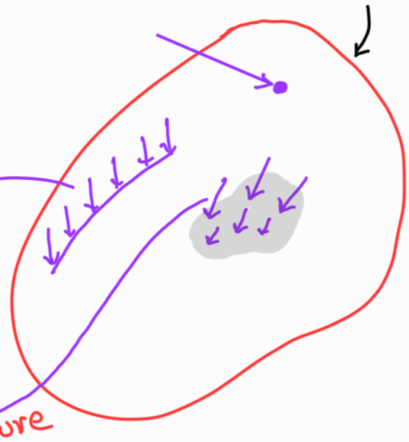
Until the last lecture we have been discussing kinematics that was to do with geometry of motion. Now we move to the next chapter of Dumir's book which lays out the precepts of dynamics that deals with analysis of the cause of motion — forces, moments

Let's review some primitive terms and concepts

FORCE : "Quantitative measure of mechanical interaction of material points/ bodies in contact or at a distance"

→ it is a vector quantity → magnitude
→ direction

3D body



Forces

- point force (idealization of force acting over a small area/vol)
- distributed force
  - → line force (force/length) → surface tension
  - → surface force (force/area) pressure
  - → volumetric force (force/vol)
    (body) ↳ e.g. gravitational force ($\rho \underline{g}$)

Contact Force vs Non-contact force (Body)

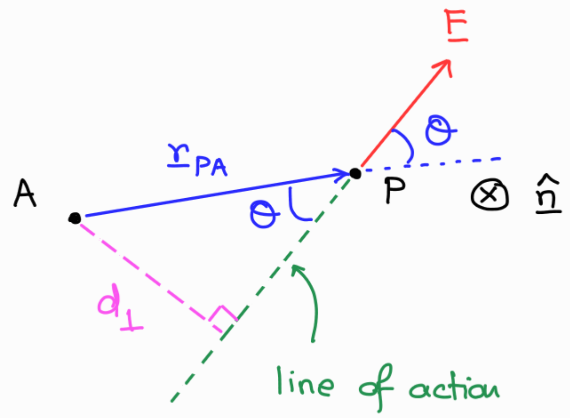
Contact Force: Pressure, surface tension, friction,

Non-contact force (Body): Gravity, Magnetic force

MOMENT of a point force about pt 'A'

Another primitive term

A force tends to push or pull, if the line of action of force passes through A. If not, then we get a rotational effect of force



$$\underline{M}_A \text{ (Moment about pt A)} = \underline{r}_{PA} \times \underline{F}$$

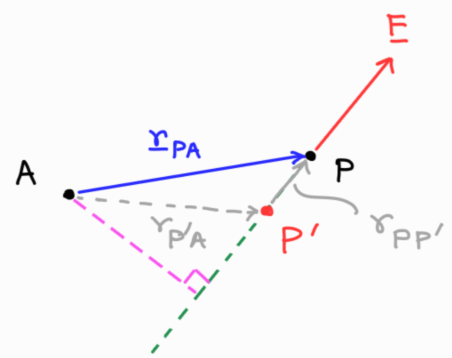
$$= |\underline{F}|\,\underbrace{|\underline{r}_{PA}|\,\sin\theta}_{d_\perp}\ \hat{\underline{n}}$$

perpendicular distance betn pt A and line of action of force $\underline{F}$

$$\Rightarrow \underline{M}_A = |\underline{F}|\, d_\perp\, \hat{\underline{n}}$$

$\hat{\underline{n}}$ is the unit normal vector to the plane formed by pt A and the line of action of force $\underline{F}$

Also, note that the moment is the same for any point P′ along the line of action of the force



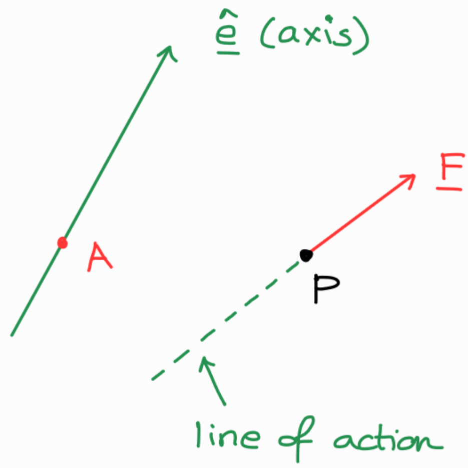
$$\underline{M}_A = \underline{r}_{PA} \times \underline{F} = (\underline{r}_{P'A} + \underline{r}_{PP'}) \times \underline{F}$$

$$= \underline{r}_{P'A} \times \underline{F}$$

$$(\because \underline{r}_{PP'} \parallel \underline{F} \Rightarrow \underline{r}_{PP'} \times \underline{F} = \underline{0})$$

Moment of forces about (along) an axis ($\hat{\underline{e}}$)

Consider the following:



(a) Choose 'P', any point on the line of action of force

(b) Choose any point 'A' on the axis along unit vector $\hat{\underline{e}}$

(c) Get $\underline{M}_A$ = moment of force $\underline{F}$ acting at 'P' abt 'A'

$$= \underline{r}_{PA} \times \underline{F}$$

(d) Obtain $\underline{M}_e$, the moment of force $\underline{F}$ acting about an axis along $\hat{\underline{e}}$

$$\boxed{\underline{M}_e = \underbrace{(\underline{M}_A \cdot \hat{\underline{e}})}_{\text{component of } \underline{M}_A \text{ along } \underline{e}} \hat{\underline{e}}}$$

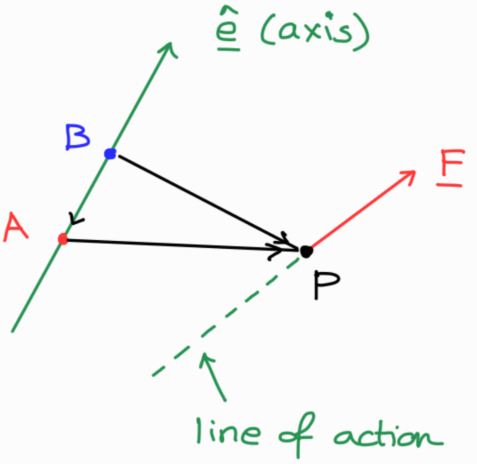
→ We have already seen that $\underline{M}_A$ does not depend on 'p' as long as 'p' is anywhere along the line of action of force

Does $\underline{M}_e$ depend on point 'A''s location along the axis ($\hat{\underline{e}}$)?

Choose another arbitrary point B on the axis ($\hat{\underline{e}}$)

$$\left|M_e'\right| = \underbrace{(\underline{r}_{PB} \times \underline{F})}_{\underline{M}_B} \cdot \underline{\hat{e}}$$

abt pt. B along e



$$= \left[(\underline{r}_{AB} + \underline{r}_{PA}) \times \underline{F} \right] \cdot \underline{\hat{e}}$$

$$= \left[(\underline{r}_{AB} \times \underline{F}) + (\underline{r}_{PA} \times \underline{F}) \right] \cdot \underline{\hat{e}}$$

$\underline{r}_{AB}$ // to $\underline{\hat{e}}$

$$\therefore \; (\underline{r}_{AB} \times \underline{F}) \cdot \underline{\hat{e}} = 0$$

$$= \underbrace{(\underline{r}_{PA} \times \underline{F})}_{\underline{M}_A} \cdot \underline{\hat{e}} = \left|\underline{M}_e\right|$$

abt pt A along axis $\underline{\hat{e}}$

$\therefore$ $\underline{M}_e$ stays the same about any point 'A' as long as 'A' lies along the axis ($\underline{\hat{e}}$)

Special cases

① Moment $\underline{M}_e = \underline{0}$ if $\underline{F}$ // $\underline{\hat{e}}$

② " $\underline{M}_e = 0$ if $\underline{F}$ intersects the axis $\underline{\hat{e}}$

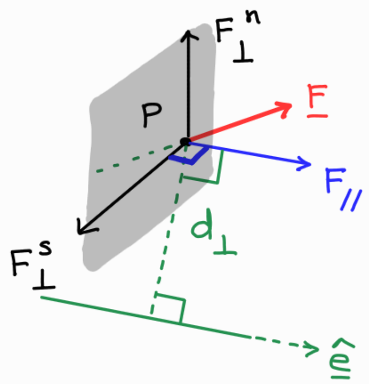
($\because$ $\underline{r}_{PA}$ // $\underline{F}$)

③ $\underline{M}_e \neq \underline{0}$ ONLY IF $\underline{F}$ and $\underline{\hat{e}}$ are skew-lines

directed lines that are neither // nor intersecting

$$\underline{F} = \underbrace{\underline{F}_{//}}_{\text{component of } \underline{F} \text{ // to } \hat{\underline{e}}} + \underbrace{\overset{\text{normal}}{\underline{F}_{\perp}^{n}} + \overset{\text{skew}}{\underline{F}_{\perp}^{s}}}_{\text{components of } \underline{F} \text{ on a plane } \perp \text{ to } \hat{\underline{e}}}$$

intersects



$$|\underline{M}_e| = |\underline{F}_{\perp}^{s}|\, d_{\perp} \quad \text{and} \quad \underline{M}_e = |\underline{M}_e|\, \hat{\underline{e}}$$

Couple : Set of forces $\underline{F}$ and $-\underline{F}$ acting along <u>different lines</u> of action

$\underline{C} \leftarrow$ symbol of couple vector

$$\underset{\text{abt pt A}}{\underline{M}_A} = \underline{r}_{PA} \times \underline{F}$$

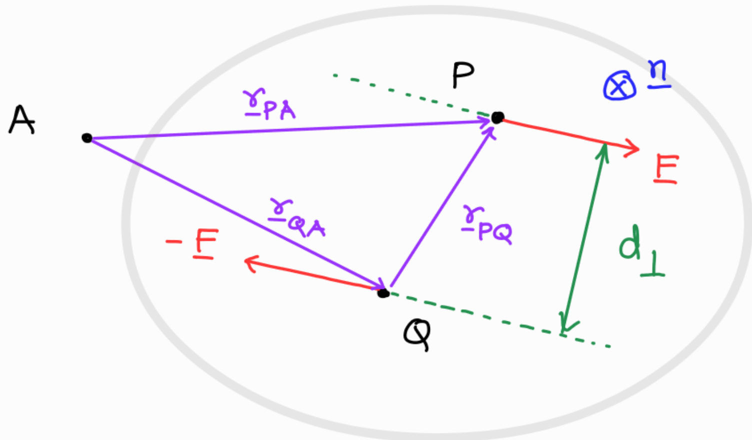
$$+ \; \underline{r}_{QA} \times (-\underline{F})$$

$$= (\underline{r}_{PA} - \underline{r}_{QA}) \times \underline{F}$$

$$= \underline{r}_{PQ} \times \underline{F}$$

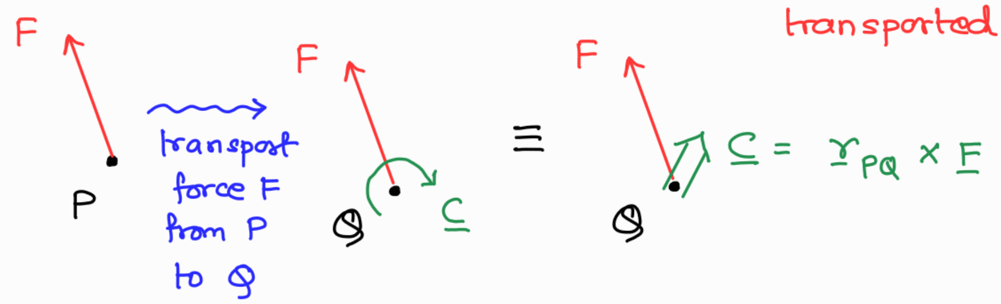
$$= |F|\, d_{\perp}\, \hat{\underline{n}} \overset{\text{def}}{\equiv} \underline{C}$$

$\rightarrow$ does <u>NOT</u> depend upon pt A (and stays the same abt <u>ANY</u> point in space)



Equivalent force-couple system

→ What happens when the pt of application of a force $\underline{F}$ gets transported to another point?



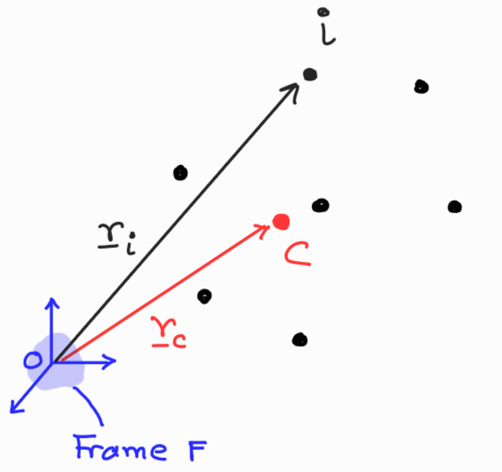
A force $\underline{F}$ acting at a point P is equivalent to a force $\underline{F}$ acting at Q and a couple $\underline{C} = \underline{r}_{PQ} \times \underline{F}$

Centre of Mass of a set of particles

Mass → measure of inertia

property of body due to which an external force is required to accelerate the body

If there are N particles, each having mass m_i and located $\underline{r}_i$ w.r.t. the csys of a ref. frame



$$\underline{r}_c = \frac{\sum_{i=1}^{N} m_i \underline{r}_i}{\sum_{i=1}^{N} m_i}$$

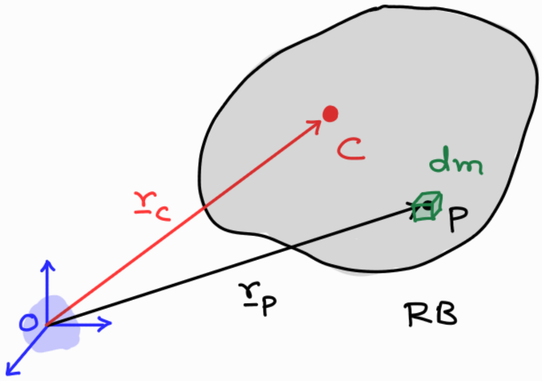
Centre of mass (COM) of a rigid body

A rigid body is a continuous distribution of point masses

Consider an infinitisimal mass dm

at any point P in RB

Suppose $\underline{r}_P = \overrightarrow{OP}$



Then point C is the centre

of mass of the rigid body (RB)

and its position vector is $\underline{r}_c$ is :

$$\underline{r}_c = \frac{\int \underline{r}_P \, dm}{\int dm} = \frac{\int_V \underline{r}_P \, \rho \, dV}{\int_V \rho \, dV}$$

density

Total mass : $m = \int dm$

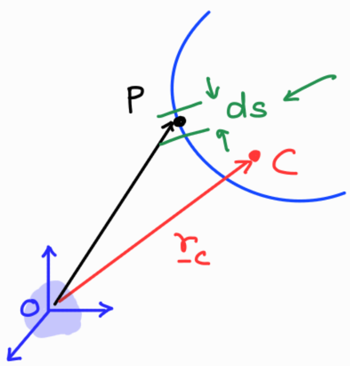
integration over the volume of the RB

If the mass of RB is approximately distributed along a curve

mass per unit length

$$\underline{r}_c = \frac{\int \underline{r}_P \, dm}{\int dm} = \frac{\int \underline{r}_P \, \lambda \, ds}{\int \lambda \, ds}$$

integration along the curve



small length element

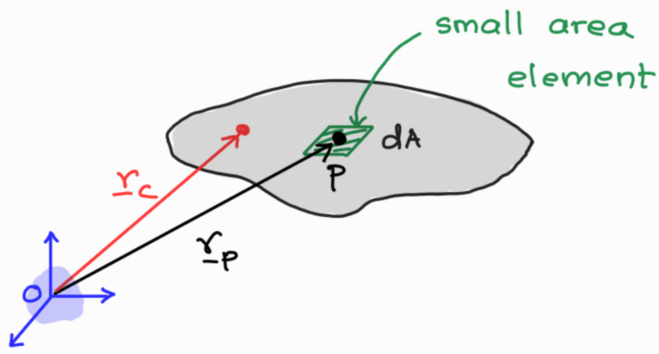
[e.g. thin rod, wire, cable]

Similarly, if the mass is distributed over an area

$$\underline{r}_c = \frac{\int \underline{r}_p \, \sigma \, dA}{\int \sigma \, dA}$$

mass per unit area

integration over the area



[e.g. thin disc/shell]

Centroid of RB: denoted by C^*

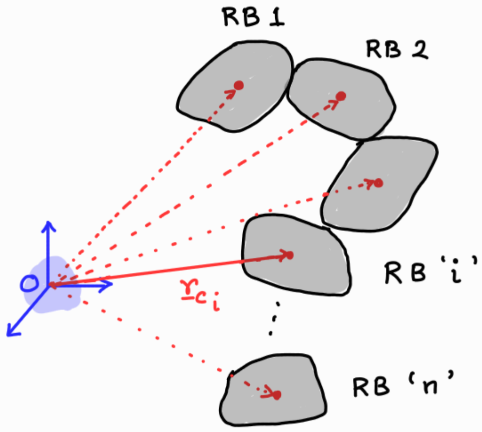
$$\underline{r}_{C^*} = \begin{cases} \dfrac{\int \underline{r}_p \, dV}{\int dV} & \text{(Volume element)} \\[2ex] \dfrac{\int \underline{r}_p \, dA}{\int dA} & \text{(Area element)} \\[2ex] \dfrac{\int \underline{r}_p \, ds}{\int ds} & \text{(Line element)} \end{cases}$$

(position vector of centroid)

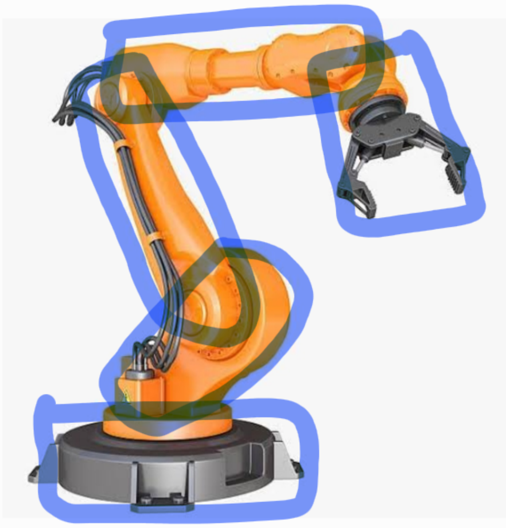
* Centroid 'C^*' and COM 'C' are the same if RB has a uniform density throughout the RB

Centre of mass of a collection of RB's

For a system of rigid bodies composed of 'n' different parts:



e.g.



Robot arm
(as an assembly of 5 RBs)

the COM is given by $\underline{r}_{cm}$

$$\underline{r}_c = \frac{\sum_{i=1}^{n} m_i \, \underline{r}_{c_i}}{\sum_{i=1}^{n} m_i}$$

position vector of COM C_i of RB 'i' measured from O

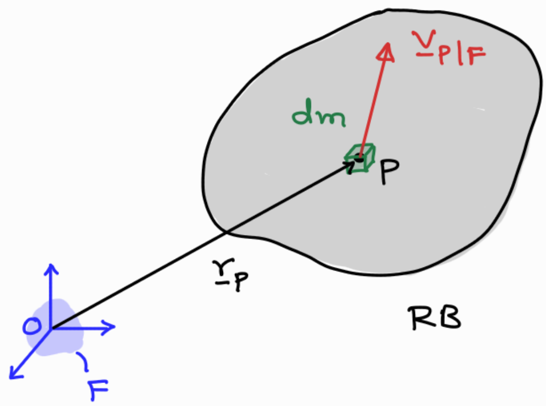
mass of RB 'i'

Momentum and Moment of Momentum of RB wrt F

1 Momentum of RB wrt frame F

also called linear momentum

Consider an elementary mass dm at any point P in RB



Suppose $\underline{v}_{P|F}$ is the velocity of point P w.r.t. the frame F

and $\underline{r}_P = \overrightarrow{OP}$

small letter 'p'

The momentum of the rigid body is denoted by $\underline{p}|_F$

and defined as:

$$\underline{p}|_F = \int \underline{v}_{P|F} \, dm = \int \left.\frac{d\underline{r}_P}{dt}\right|_F dm$$

integration over entire RB

$$= \frac{d}{dt}\left(\int \underline{r}_P \, dm\right)\Bigg|_F$$

[Using Leibnitz integral formula]

Both velocity and mass distributions must be known for integ.

total mass

$$= \frac{d}{dt}\left(m\, \underline{r}_c\right)\Big|_F$$

$$= m \left.\frac{d}{dt}\underline{r}_c\right|_F$$

$$\underline{r}_c = \frac{\int \underline{r}_P \, dm}{\int dm}$$

$$\Rightarrow m\, \underline{r}_c = \int \underline{r}_P \, dm$$

$$= m\, \underline{v}_{c|F}$$ = momentum of RB's Centre of mass (COM)

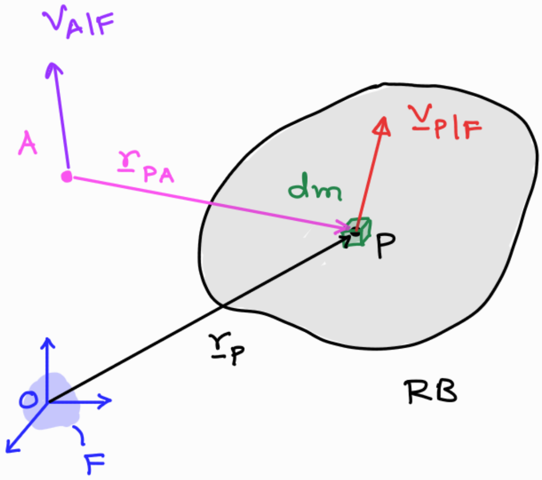
For a system of RBs, the momentum would be

$$\underline{p}\big|_F = \sum_{i=1}^{n} m_i \, \underline{v}_{c_i|F}$$

② Moment of Momentum of an RB about a pt A and wrt frame 'F'

$\underbrace{\text{Moment of Momentum}}$ popular as ANGULAR MOMENTUM

The angular momentum of an RB about a point A is denoted by $\underline{H}_A\big|_F$ and defined as:



$$\underline{H}_A\big|_F = \int \left(\underline{r}_{PA} \times \underline{v}_{PA|F} \, dm \right)$$

$$\underline{v}_{PA|F} = \underline{v}_{P|F} - \underline{v}_{A|F}$$

↑ relative velocity of point P w.r.t point A compared in F

'A' can be any point in space

$$\text{'A'} = \begin{cases} \text{material point of RB (or)} \\ \text{point on fixed frame of ref F (or)} \\ \text{point on a RB that is moving w.r.t F} \end{cases}$$

EULER'S AXIOMS governing dynamics of RBs

$\underbrace{\text{AXIOMS}}$

1. foundational statements or principles that are accepted without proofs
2. they are building blocks of a bigger logical reasoning

There exists a reference frame 'I', with a point 'O' fixed in 'I' (serving as the origin of a csys attached to 'I'), such that, for any system, if the following two axioms: (a) linear momentum balance, and (b) angular momentum balance are satisfied then the frame of reference is called an Inertial frame of reference!

Axiom (1): Linear momentum balance

The time rate of change of the momentum of the COM of RB in inertial frame is equal to the net external force acting on RB

$$\Rightarrow \quad \frac{d}{dt}\{\underbrace{\underline{p}|_I}_{\substack{\text{momentum of} \\ \text{the RB wrt frame 'I'}}}\}\Big|_I = \dot{\underline{p}}\Big|_I = \underbrace{\underline{F}^{ext}}_{\substack{\text{net external} \\ \text{force acting} \\ \text{on the RB}}} \qquad \text{— (I)}$$

$$\Rightarrow \quad \frac{d}{dt}\left(m\,\underline{v}_{c|I}\right)\Big|_I = \underline{F}^{ext}$$

$$\Rightarrow \quad m\,\underline{a}_{c|I} = \underline{F}^{ext}$$

Equivalently, the first Euler's axiom is the product of RB's mass times the acceleration of its COM in inertial frame 'I'

$$\underline{F}_{net}^{ext} = m\,\underline{a}_{c|I} = m\,\dot{\underline{v}}_{c|I} = m\,\ddot{\underline{r}}_{c|I}$$

Thus, using the first axiom, we can describe the motion of centre of mass of an RB.

Recall that motion of a material point of an RB may be described using different coordinate systems:

(a) Cartesian csys

$$\underline{F}_{net}^{ext} = F_1\,\hat{\underline{e}}_1 + F_2\,\hat{\underline{e}}_2 + F_3\,\hat{\underline{e}}_3 = m\left(\ddot{x}_1\,\hat{\underline{e}}_1 + \ddot{x}_2\,\hat{\underline{e}}_2 + \ddot{x}_3\,\hat{\underline{e}}_3\right)$$

(b) Cylindrical csys

$$\underline{F}_{net}^{ext} = F_r\,\hat{\underline{e}}_r + F_\phi\,\hat{\underline{e}}_\phi + F_z\,\hat{\underline{e}}_z$$

$$= m\left[\left(\ddot{r} - r\dot{\phi}^2\right)\hat{\underline{e}}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)\hat{\underline{e}}_\phi + \ddot{z}\,\hat{\underline{e}}_z\right]$$

(c) Path csys

$$\underline{F}_{net}^{ext} = F_t\,\hat{\underline{e}}_t + F_n\,\hat{\underline{e}}_n = m\left(\ddot{s}\,\hat{\underline{e}}_t + \frac{\dot{s}^2}{\rho}\,e_n\right)$$

Note that there can be no force component F_b normal to the osculating plane.

Axiom 2 : Angular momentum balance

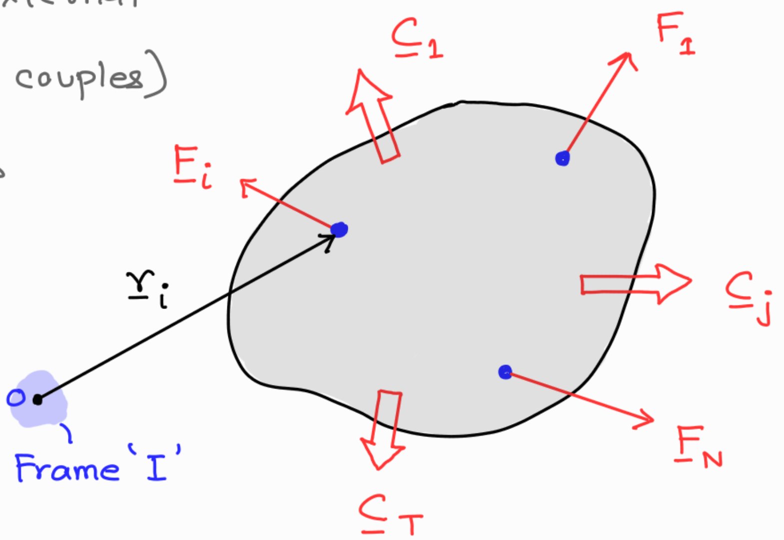
The time rate of change of angular momentum of RB about a specified reference point 'O' in an inertial frame 'I' is equal to the sum of all moments/torques caused by external loads

$$\text{(II)} \quad \text{———} \quad \frac{d}{dt}\{\underbrace{\underline{H}_{O|I}}_{\substack{\text{angular momentum} \\ \text{of RB wrt pt O fixed} \\ \text{in frame 'I'}}}\}\Big|_I = \dot{\underline{H}}_{O|I} = \underbrace{\underline{M}_{O,net}^{ext}}_{\substack{\text{net moment} \\ \text{due to all ext.} \\ \text{loads about 'O'}}}$$

Eqns (I) and (II) leads to SIX scalar equations.

What do we mean by net external forces and moments/torques?

In general, there may be N external forces (excluding forces causing couples) and T torques/couples of forces acting on an RB



Net external force

$$\underline{F}^{ext} = \sum_{i=1}^{N} \underline{F}_i$$

Net external moment abt pt 'O'

$$\underline{M}_{O,net}^{ext} = \sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i + \sum_{j=1}^{T} \underline{C}_j$$

$\uparrow$ position vector of the point of application of $\underline{F}_i$ on RB

Inertial frames of reference

Frames of reference which satisfies the two Euler's axioms are called inertial frame of reference.

In simple words, an inertial frame is a reference frame relative to which a uniform motion ($V_{P|F}$ = constant) can be sustained without force. Therefore, any disturbance of a RB from a state of rest or uniform motion w.r.t. inertial frame can occur only in response to unbalanced forces and moments, acting separately or together.

Now the question is what physical ref. frame (or RB) in the real world can be treated as a reference frame?

Plainly, a fixed reference frame/body (or a ref. frame at rest) is an inertial frame. But a frame/body can be identified as fixed in space only relative to other bodies known to be fixed in space. So the idea of an inertial frame being fixed in space is meaningless.

It is impossible to find an exact physical inertial ref. frame!

Our most natural choice appears to be the Earth frame. However, we know that the Earth rotates and revolves and therefore, can we use Earth frame as an inertial frame?

We cannot use the Earth frame as an inertial frame for dynamics of astronomical bodies, however, we could approximately use it for bodies for which Earth's movement does not affect the calculations very much.

Summary of Euler's Axioms

Euler's axioms related forces and moments that act on a body to its translational and rotational motions.

Axiom 1 : Forces $\leftrightarrow$ translational motion

$$\frac{d}{dt}\left\{ \underline{P}|_{I} \right\}\Big|_{I} = \underline{F}^{ext}_{net}$$

Axiom 2: Moments / Torques $\leftrightarrow$ rotational motion

$$\frac{d}{dt}\left\{ H_{O|I} \right\}\Big|_{I} = \underline{M}^{ext}_{o,\,net}$$

Any non-rotating, uniformly translating frame can be treated as an inertial reference frame I, as it would satisfy the two Euler's axioms. In real world, a preferred frame may be identified as any frame which has at most a uniform translational velocity relative to (say) the astronomical frame fixed in distant stars.