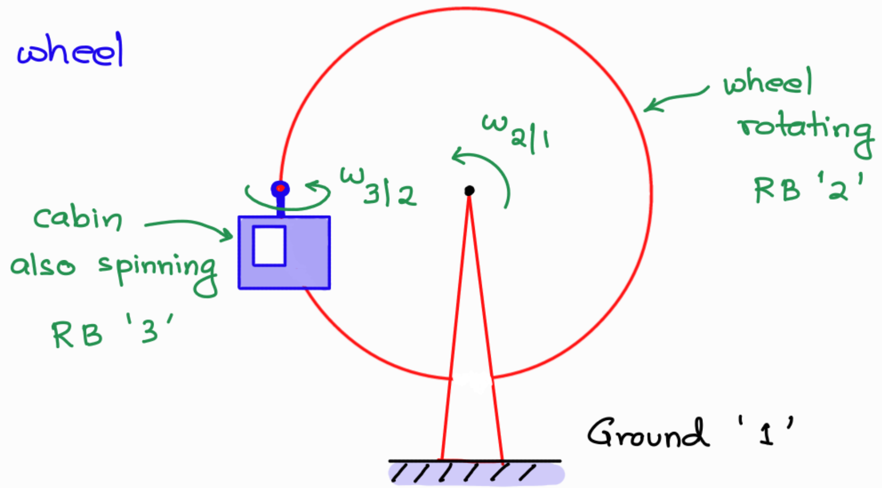
Multiple Reference Frames

In mechanics, multiple reference frames are frequently used to analyze systems where different parts of a problem are moving relative to each other. Use of multiple reference frames allow engineers to simplify analysis of complex systems

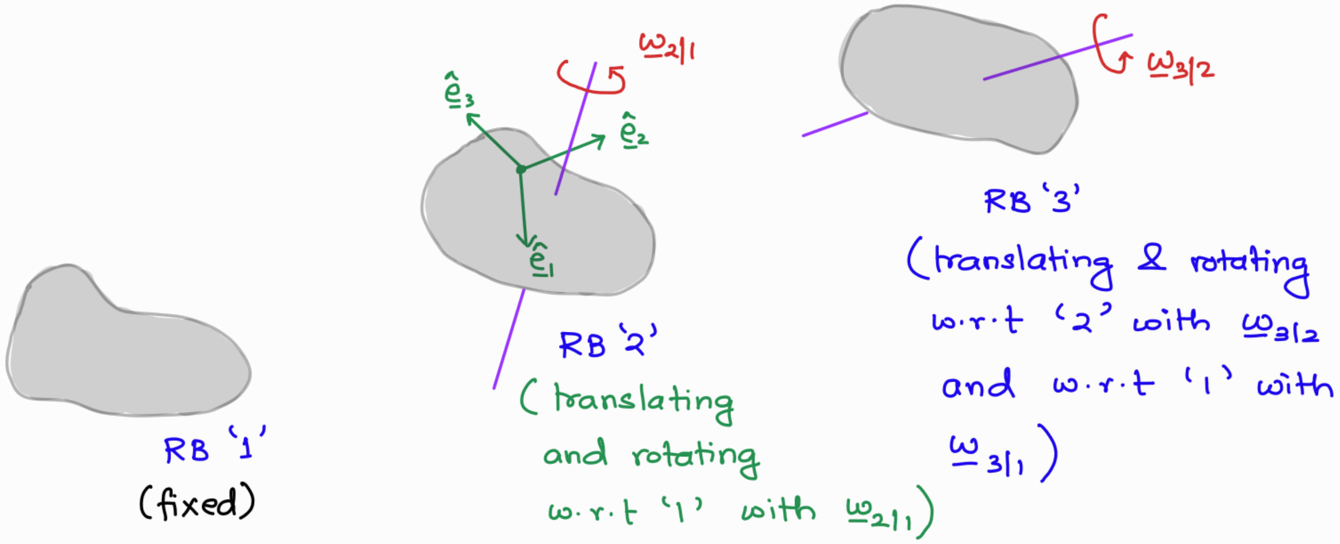
e.g. Ferris wheel



Relating the movement of a point w.r.t different reference frames is called composition of movements

Composition of angular velocities

Consider three reference frames



If we consider a vector $\underline{A}$ changing its position with time, then we can compose the velocity vector of $\underline{A}$ using different frames as follows:

Frames 1 and 2 : $$\left.\frac{d\underline{A}}{dt}\right|_1 = \left.\frac{d\underline{A}}{dt}\right|_2 + \underline{\omega}_{2|1} \times \underline{A} \qquad \text{— ①}$$

Frames 2 and 3 : $$\left.\frac{d\underline{A}}{dt}\right|_2 = \left.\frac{d\underline{A}}{dt}\right|_3 + \underline{\omega}_{3|2} \times \underline{A} \qquad \text{— ②}$$

Frames 1 and 3 : $$\left.\frac{d\underline{A}}{dt}\right|_1 = \left.\frac{d\underline{A}}{dt}\right|_3 + \underline{\omega}_{3|1} \times \underline{A} \qquad \text{— ③}$$

Using ② in ① : $$\left.\frac{d\underline{A}}{dt}\right|_1 = \left(\left.\frac{d\underline{A}}{dt}\right|_3 + \underline{\omega}_{3|2} \times \underline{A}\right) + \underline{\omega}_{2|1} \times \underline{A} \qquad \text{— ④}$$

Now equate ③ and ④ :

$$\cancel{\left.\frac{d\underline{A}}{dt}\right|_3} + \underline{\omega}_{3|1} \times \underline{A} = \cancel{\left.\frac{d\underline{A}}{dt}\right|_3} + \underline{\omega}_{3|2} \times A + \underline{\omega}_{2|1} \times \underline{A}$$

$$\Rightarrow \quad \underline{\omega}_{3|1} \times \underline{A} = \left(\underline{\omega}_{3|2} + \underline{\omega}_{2|1}\right) \times \underline{A}$$

$$\Rightarrow \quad \boxed{\underline{\omega}_{3|1} = \underline{\omega}_{3|2} + \underline{\omega}_{2|1}} \qquad (\because \ \underline{A} \text{ is arbitrary})$$

Composition of angular velocities

This result brings out the additive property of angular velocities that is, if you know two of them, you can compute the third!

Also, note that if $1 = 3 =$ fixed, $\cancelto{0}{\underline{\omega}_{3|1}} = \underline{\omega}_{\cancelto{1}{3}|2} + \underline{\omega}_{2|1} = \underline{\omega}_{1|2} + \underline{\omega}_{2|1}$

$$\Rightarrow \quad \boxed{\underline{\omega}_{1|2} = -\underline{\omega}_{2|1}}$$

Composition of angular accelerations

Having observed that $\underline{\omega}_{3|1} = \underline{\omega}_{3|2} + \underline{\omega}_{2|1}$, we now differentiate the result w.r.t. time in frame 1:

$$\frac{d\,\underline{\omega}_{3|1}}{dt}\bigg|_1 = \frac{d\,\underline{\omega}_{3|2}}{dt}\bigg|_1 + \frac{d\,\underline{\omega}_{2|1}}{dt}\bigg|_1$$

$$\Rightarrow \dot{\underline{\omega}}_{3|1} = \underbrace{\dot{\underline{\omega}}_{3|2}\Big|_1}_{\left[\dot{\underline{\omega}}_{3|2}\big|_2 + \underline{\omega}_{2|1}\times\underline{\omega}_{3|2}\right]} + \dot{\underline{\omega}}_{2|1}$$

$$\Rightarrow \dot{\underline{\omega}}_{3|1} = \dot{\underline{\omega}}_{3|2} + \underline{\omega}_{2|1}\times\underline{\omega}_{3|2} + \dot{\underline{\omega}}_{2|1}$$

Note:

① $\dot{\underline{\omega}}_{3|1}\big|_1 = \dot{\underline{\omega}}_{3|1}\big|_3 = \dot{\underline{\omega}}_{3|1}$

② $\dot{\underline{\omega}}_{2|1}\big|_1 = \dot{\underline{\omega}}_{2|1}\big|_2 = \dot{\underline{\omega}}_{2|1}$

③ $\dot{\underline{\omega}}_{3|2}\big|_1$ ← has to be found

④ $\dot{\underline{\omega}}_{3|2}\big|_2 = \dot{\underline{\omega}}_{3|2}$

⑤ $\dot{\underline{A}}\big|_F = \dot{\underline{A}}\big|_m + \underline{\omega}_{m|F}\times\underline{A}$

$$\boxed{\dot{\underline{\omega}}_{3|1} = \dot{\underline{\omega}}_{3|2} + \dot{\underline{\omega}}_{2|1} + \underline{\omega}_{2|1}\times\underline{\omega}_{3|2}}$$

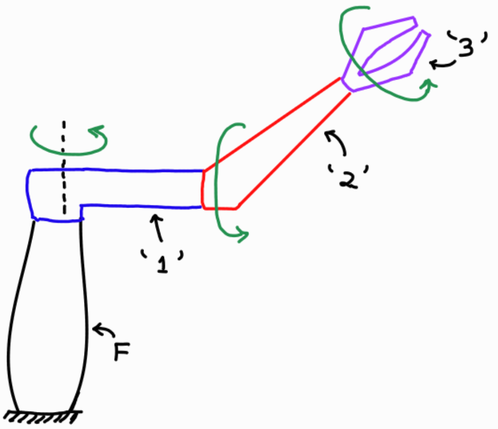
Composition of angular accelerations

In general, $\dot{\underline{\omega}}_{3|1} \neq \dot{\underline{\omega}}_{3|2} + \dot{\underline{\omega}}_{2|1}$, that is the additivity property that holds for composition of angular velocities does not apply for composition of angular acceleration

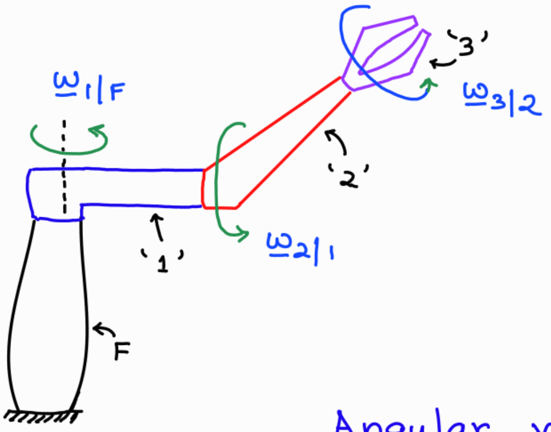
Consider, for example, a connected system of bodies (e.g. robotic arm) ungergoing "general motion"

⇓

meaning NOT pure translation



Robotic arm



$\underline{\omega}_{1|F}$, $\underline{\omega}_{2|1}$, $\underline{\omega}_{3|2}$ ⇐ Baseline angular velocities

Now, we will compose the angular velocity and acceleration

Angular velocity

$$\underline{\omega}_{3|F} = \underline{\omega}_{3|2} + \underline{\omega}_{2|1} + \underline{\omega}_{1|F}$$ (by additive prop)

Angular acceleration

$$\dot{\underline{\omega}}_{3|F} = \overset{\cdot}{\overline{\underline{\omega}_{3|2}}}\Big|_F + \overset{\cdot}{\overline{\underline{\omega}_{2|1}}}\Big|_F + \overset{\cdot}{\overline{\omega_{1|F}}}\Big|_F$$

$$= \dot{\underline{\omega}}_{3|2} + \underline{\omega}_{2|F} \times \underline{\omega}_{3|2}$$

$$+ \dot{\underline{\omega}}_{2|1} + \underline{\omega}_{1|F} \times \underline{\omega}_{2|1}$$

$$+ \dot{\underline{\omega}}_{1|F}$$

$$= \dot{\underline{\omega}}_{3|2} + (\underline{\omega}_{2|1} + \underline{\omega}_{1|F}) \times \underline{\omega}_{3|2}$$

$$+ \dot{\underline{\omega}}_{2|1} + \underline{\omega}_{1|F} \times \underline{\omega}_{2|1}$$

$$+ \dot{\underline{\omega}}_{1|F}$$

$$\overset{\cdot}{\overline{\underline{\omega}_{3|2}}}\Big|_F = \underbrace{\dot{\underline{\omega}}_{3|2}\Big|_2}_{\dot{\underline{\omega}}_{3|2}} + \underline{\omega}_{2|F} \times \underline{\omega}_{3|2}$$

$$\overset{\cdot}{\overline{\underline{\omega}_{2|1}}}\Big|_F = \underbrace{\dot{\underline{\omega}}_{2|1}\Big|_1}_{\dot{\underline{\omega}}_{2|1}} + \underline{\omega}_{1|F} \times \underline{\omega}_{2|1}$$

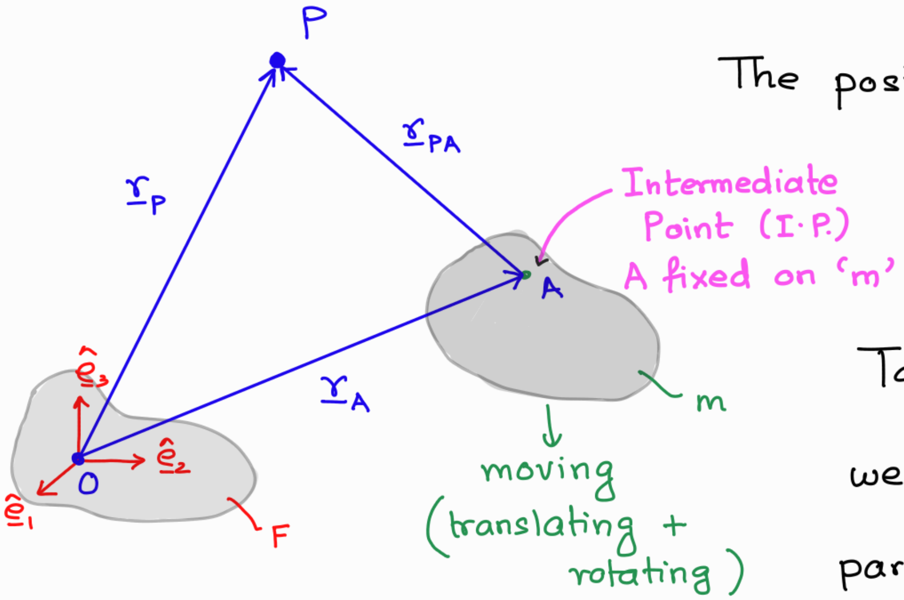
$$\dot{\underline{\omega}}_{1|F}\Big|_F = \dot{\underline{\omega}}_{1|F}$$

$$\underline{\omega}_{2|F} = \underline{\omega}_{2|1} + \underline{\omega}_{1|F}$$

In kinematics, it is interesting to relate the movement of the same object — for example, a vehicle — in two different reference frames, say, the earth and a second vehicle. Since the time derivative of position vector is reference-frame dependent, we must investigate the relationship between the time derivatives of the same vector $\underline{A}$ in different reference frames

Particle Kinematics with multiple reference frames

We are now in a position to describe the motion of a particle P where more than one reference frames may be employed. In particular, we will examine motion of P w.r.t. fixed frame 'F' and moving frame 'm'.



The position vector of particle P

$$\underline{r}_P = \underline{r}_A + \underline{r}_{PA}$$

Taking derivative w.r.t. time we get the velocity vector of particle P :

$$v_P\big|_F = \frac{d\underline{r}_P}{dt}\bigg|_F = \underbrace{\frac{d}{dt}\underline{r}_A\bigg|_F} + \underbrace{\frac{d}{dt}\underline{r}_{PA}\bigg|_F}$$

$$= \underline{v}_{A|F} + \left(\underbrace{\frac{d\underline{r}_{PA}}{dt}\bigg|_m}_{\underline{v}_{P|m}} + \underline{\omega}_{m|F} \times \underline{r}_{PA}\right)$$

$$\boxed{\underline{v}_{P|F} = v_{A|F} + v_{P|m} + \underline{\omega}_{m|F} \times \underline{r}_{PA}}$$

To obtain the acceleration vector, we differentiate w.r.t. time again:

$$\underline{a}_{P|F} = \frac{d}{dt}\underline{v}_{P|F}\bigg|_F = \underbrace{\frac{d}{dt}\underline{v}_{A|F}\bigg|_F}_{①} + \underbrace{\frac{d}{dt}\underline{v}_{P|m}\bigg|_F}_{②} + \underbrace{\frac{d}{dt}\left(\underline{\omega}_{m|F} \times \underline{r}_{PA}\right)}_{③}$$

Term ① : $\frac{d}{dt}\underline{v}_{A|F}\Big|_F = \underline{a}_{A|F}$

Term ② : $\frac{d}{dt}\underline{v}_{P|m}\Big|_F = \underbrace{\frac{d}{dt}(\underline{v}_{P|m})\Big|_m}_{} + \underline{\omega}_{m|F}\times\underline{v}_{P|m}$

$$= \underline{a}_{P|m} + \underline{\omega}_{m|F}\times\underline{v}_{P|m}$$

Term ③ : $\frac{d}{dt}(\underline{\omega}_{m|F}\times\underline{r}_{PA})\Big|_F = \left(\underbrace{\frac{d}{dt}\underline{\omega}_{m|F}\Big|_F}_{\dot{\underline{\omega}}_{m|F}}\times\underline{r}_{PA}\right) + \underbrace{\left(\underline{\omega}_{m|F}\times\frac{d\underline{r}_{PA}}{dt}\Big|_F\right)}_{}$

$$\underline{\omega}_{m|F}\times\frac{d\underline{r}_{PA}}{dt}\Big|_F = \underline{\omega}_{m|F}\times\left(\frac{d\underline{r}_{PA}}{dt}\Big|_m + \underline{\omega}_{m|F}\times\underline{r}_{PA}\right)$$

$$= \underline{\omega}_{m|F}\times\left(\underline{v}_{P|m} + \underline{\omega}_{m|F}\times\underline{r}_{PA}\right)$$

Thus,

$$\underbrace{\underline{a}_{P|F}}_{\text{Acc of P meas wrt fixed frame 'F'}} = \underbrace{\underline{a}_{A|F}}_{\text{Acc of origin A of moving frame wrt fixed frame}} + \underbrace{\underline{a}_{P|m}}_{\text{Acc of P meas. wrt moving frame}} + \underbrace{\left(\overset{\text{angular acc. }\underline{\alpha}_{m|F}}{\dot{\underline{\omega}}}\times\underline{r}_{PA}\right)}_{\text{Tangential component}} + \underbrace{\overset{\underline{\omega}_{m|F}}{\underline{\omega}}\times(\underline{\omega}\times\underline{r}_{PA})}_{\text{Centripetal/Normal component}} + \underbrace{2(\underline{\omega}\times\underline{v}_{P|m})}_{\text{CORIOLIS Acc.}}$$

The acc. term $2(\underline{\omega}_{m|F}\times\underline{v}_{P|m})$ is called the Coriolis acceleration.

Note that the Coriolis acceleration appears ONLY IF :

① $\underline{\omega}_{m|F} \neq 0$ (No rotation of moving frame 'm' w.r.t 'F' will cause NO Coriolis acceleration!)

AND

② $\underline{v}_{P|m} \neq 0$ (Having a rotating frame 'm' wrt 'F' is not enough. The particle P must also have a non-zero relative velocity wrt rotating frame 'm')

AND

③ $\underline{v}_{P|m}$ is not parallel to $\underline{\omega}_{m|F}$

(In addition to requirements ① and ②, the non-zero translational velocity relative to moving frame 'm' must have a non-zero component perpendicular to the angular velocity of moving frame 'm')

Interpretation of the acceleration terms

$\underline{a}_{P|F}$	acceleration of P measured from fixed ref. frame 'F'	motion of P observed from frame 'F'
$\underline{a}_{P|m}$	acceleration of P measured from moving ref. frame 'm'	motion of P observed from frame 'm'
	+	
$\underline{a}_{A|F}$	acceleration of moving ref. frame 'm' w.r.t. 'F'	motion of moving frame 'm' observed from the fixed ref. frame 'F'
	+	
$\dot{\underline{\omega}} \times \underline{r}_{PA}$	tangential acceleration of P caused by rotation of moving ref. frame 'm'	
	+	
$\underline{\omega} \times (\underline{\omega} \times \underline{r}_{PA})$	centripetal acceleration directed radially inward toward the axis of rotation caused by rotation of moving frame 'm'	
	+	
$2(\underline{\omega} \times \underline{v}_{P|m})$	Coriolis acceleration due to combined effect of P moving relative to 'm' & rotation of 'm'	interacting motion

Some special cases :

Case 1 : Moving frame 'm' is purely TRANSLATING w.r.t. 'F'

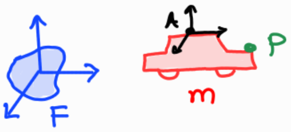
$$\Rightarrow \quad \underline{\omega}_{m|F} = \underline{0} \quad \left(\text{and} \quad \underline{\alpha}_{m|F} = \dot{\underline{\omega}}_{m|F} = 0\right)$$

$$\Rightarrow \quad \boxed{\underline{v}_{P|F} = \underline{v}_{P|m} + \underline{v}_{A|F}} \quad \text{and} \quad \boxed{\underline{a}_{P|F} = \underline{a}_{P|m} + \underline{a}_{A|F}}$$

Additionally, if particle P is a part of the moving frame 'm' then :

$$\underline{v}_{P|m} = 0 \qquad \bigg| \qquad \underline{a}_{P|m} = 0$$

$$\Rightarrow \underline{v}_{P|F} = \underline{v}_{A|F} \qquad \bigg| \qquad \Rightarrow \underline{a}_{P|F} = \underline{a}_{A|F}$$



Case 2 : Particle P is a part of moving frame 'm' but the moving frame is both translating and rotating $(\underline{\omega}_{m|F} \neq \underline{0})$
$(\underline{\alpha}_{m|F} \neq \underline{0})$

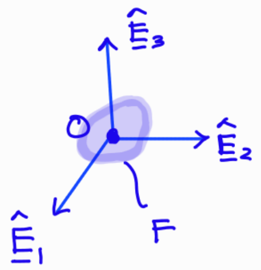
$$\Rightarrow \quad \underline{v}_{P|m} = \underline{0} \quad \text{and} \quad \underline{a}_{P|m} = \underline{0}$$

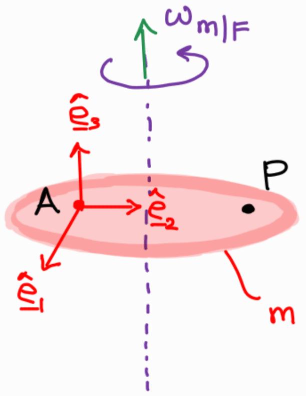
$$\therefore \quad \underline{v}_{P|F} = \underline{v}_{A|F} + \left(\underline{\omega}_{m|F} \times \underline{r}_{PA}\right)$$

and $$\underline{a}_{P|F} = \underline{a}_{A|F} + \left(\dot{\underline{\omega}}_{m|F} \times r_{PA}\right) + \underline{\omega}_{m|F} \times \left(\underline{\omega}_{m|F} \times \underline{r}_{PA}\right)$$

no Coriolis acc. !

Case 3 : Suppose origin A and particle P are on the same plane perpendicular to the angular velocity vector $\underline{\omega}_{m|F}$ of frame 'm' i.e. 'm' is rotating such that $\underset{(m)}{\hat{\underline{e}}_3} \parallel \underset{(F)}{\hat{\underline{E}}_3}$





$\underline{r}_{PA} \perp \underline{\omega}_{m|F}$

Then for such a case

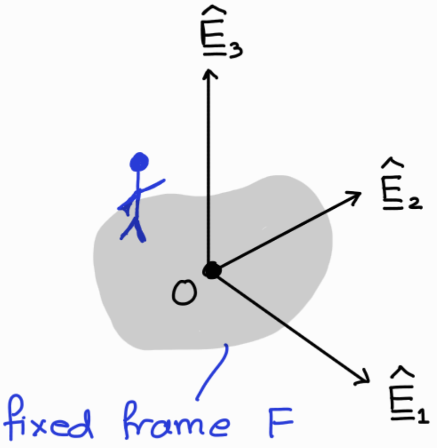
$$\underline{\omega} \times (\underline{\omega} \times \underline{r}_{PA})$$

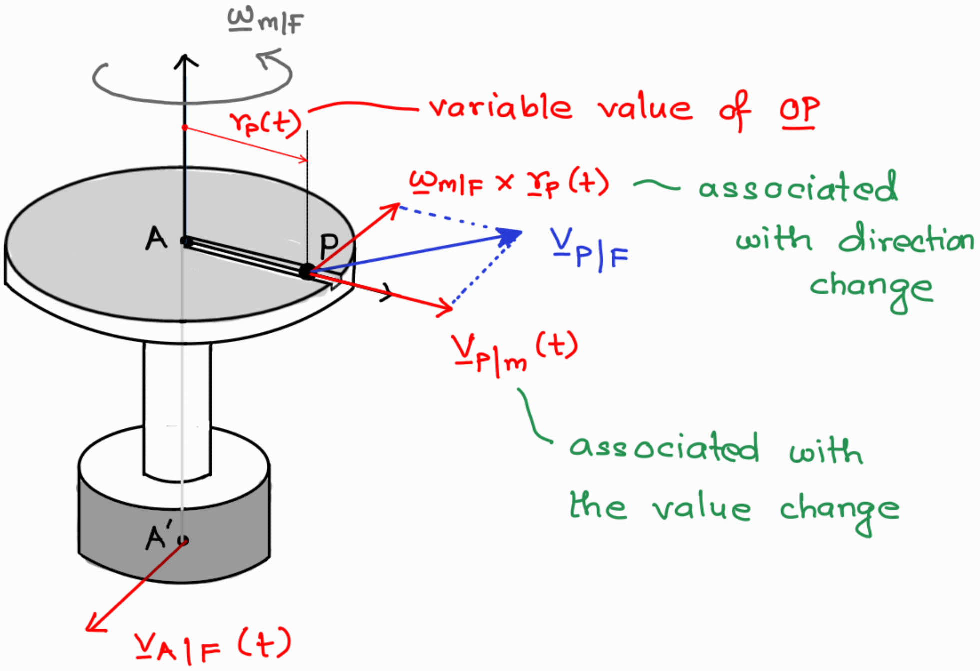
$$= \cancelto{0}{(\underline{\omega} \cdot \underline{r}_{PA})}\, \underline{\omega} - (\underline{\omega} \cdot \underline{\omega})\, \underline{r}_{PA}$$

$$= \underbrace{-\,\omega^2\, \underline{r}_{PA}}_{\text{Centripetal acceleration directed inwards from P to A}}$$

Lets consider the example of moving particle P on rotating frame 'm' wrt 'F'

Velocity of P





Acceleration vector of P

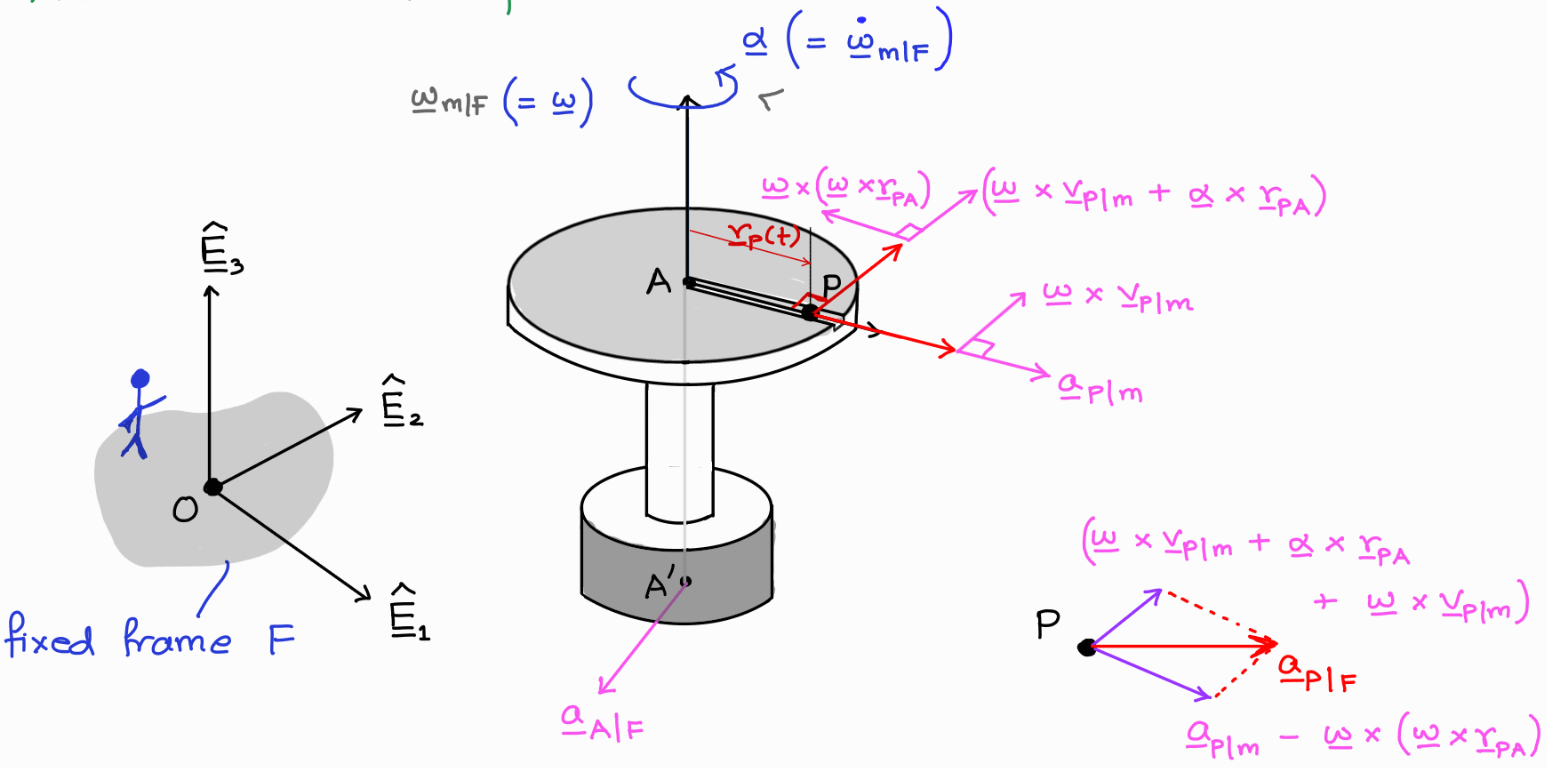


Illustration of composition of acceleration (assuming $\underline{a}_{A|F} = 0$)