

Recap

In the last lecture, we had studied the time derivatives of the position vector (i.e. velocity and acceleration) of a particle P w.r.t different coordinate systems of a ref. frame.

In kinematics, it is interesting to relate the motion of the same body (modelled as a particle or an RB) in two different reference frames.

Since the time-derivative of  vectors is reference-frame-dependent we will discuss the relationship between the time derivatives of the same vector $\underline{A}$ in different reference frames 'F' and 'm'. You will see that the derivatives are related through:

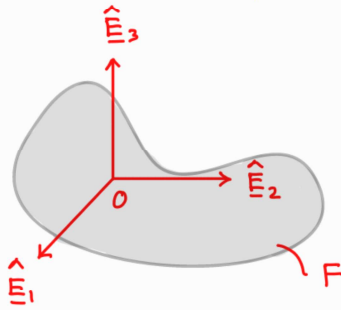
$$\left. \frac{d\underline{A}}{dt} \right|_F = \left. \frac{d\underline{A}}{dt} \right|_m + \underbrace{\underline{\omega}_{m|F}}_{\substack{\text{angular} \\ \text{velocity of} \\ \text{moving ref. 'm'} \\ \text{w.r.t ref 'F'}}} \times \underline{A} \quad \left. \vphantom{\frac{d\underline{A}}{dt}} \right\} \begin{array}{l} \text{will define} \\ \text{it formally} \\ \text{next} \end{array}$$

You will see that the time derivatives in the two different ref. frames differ only if they are rotating relative to each other.

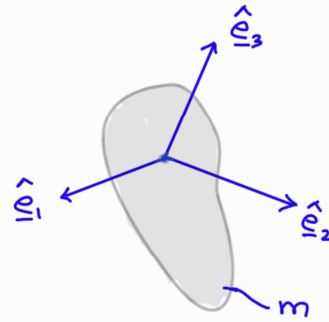
Angular velocity vector of a rotating ref. frame 'm'

w.r.t fixed frame F

Consider a fixed frame RB 'F' and a rotating frame of reference: RB 'm'



Fixed frame (F)



Rotating frame (m)

$$\left. \frac{d\hat{E}_i}{dt} \right|_F \stackrel{\text{w.r.t}}{=} \underline{0}$$

$$\left. \frac{d\hat{e}_i}{dt} \right|_m \stackrel{\text{w.r.t}}{=} \underline{0}$$

$\underline{\omega}_m|_F$: angular velocity vector
of rigid body (RB) 'm'
w.r.t RB 'F'

measures the rate of change
→ of orientation of RB 'm' w.r.t.
RB 'F'

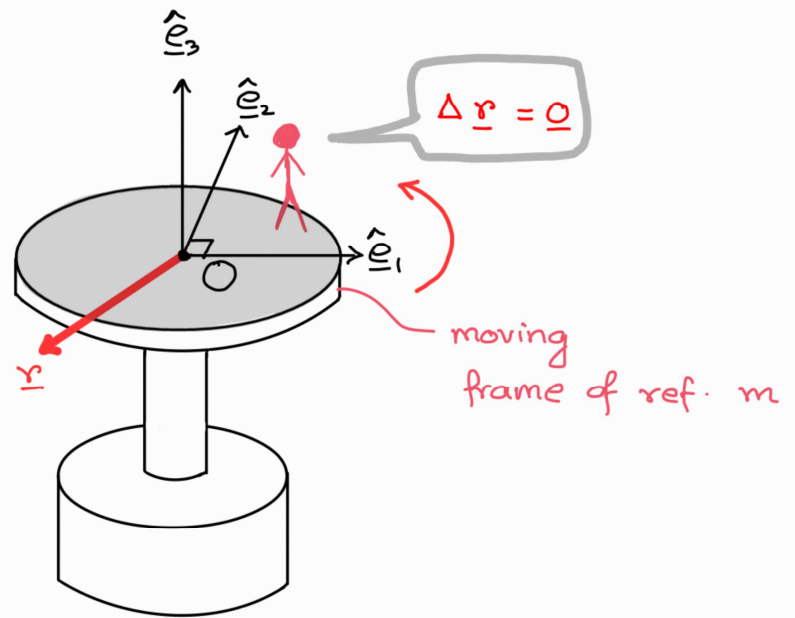
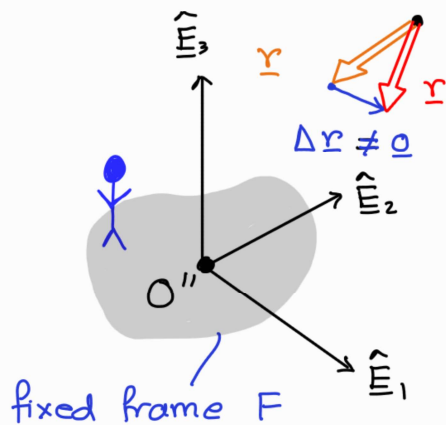
How do we measure orientation?



Let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ be three mutually perpendicular vectors embedded in RB 'm', and $\hat{E}_1, \hat{E}_2, \hat{E}_3$ are three m.p vectors embedded in RB 'F', then if the orientation of the triad $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$ appears to change to an observer tied to RB 'F', then

$$\underline{\omega}_m|_F \neq \underline{0}$$

Here is a more physical example



Note that $\left. \frac{d\hat{E}_1}{dt} \right|_F = \left. \frac{d\hat{E}_2}{dt} \right|_F = \left. \frac{d\hat{E}_3}{dt} \right|_F = \underline{0}$

Also, $\left. \frac{d\hat{e}_1}{dt} \right|_m = \left. \frac{d\hat{e}_2}{dt} \right|_m = \left. \frac{d\hat{e}_3}{dt} \right|_m = \underline{0}$

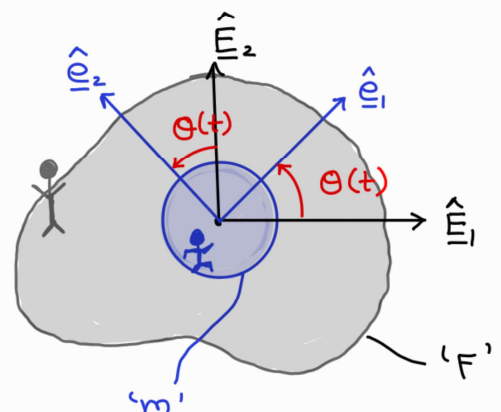
but if any one of the following is non-zero,

$$\left. \frac{d\hat{e}_1}{dt} \right|_F, \left. \frac{d\hat{e}_2}{dt} \right|_F, \left. \frac{d\hat{e}_3}{dt} \right|_F \neq \underline{0}$$

then $\underline{\omega}_m|_F \neq \underline{0}$

Let us consider a simple case of planar motion of rotating frame 'm' w.r.t. fixed ref. frame 'F'

Every point/particle fixed in 'm' moves in a plane, hence called planar motion



$$\hat{\underline{e}}_1 = \cos \theta \hat{\underline{E}}_1 + \sin \theta \hat{\underline{E}}_2$$

$$\hat{\underline{e}}_2 = -\sin \theta \hat{\underline{E}}_1 + \cos \theta \hat{\underline{E}}_2$$

$$\left. \frac{d\hat{\underline{e}}_1}{dt} \right|_F = \left. \frac{d}{dt} (\cos \theta \hat{\underline{E}}_1 + \sin \theta \hat{\underline{E}}_2) \right|_F$$

$$= (-\sin \theta) \dot{\theta} \hat{\underline{E}}_1 + (\cos \theta) \dot{\theta} \hat{\underline{E}}_2$$

manipulate
using cross
product

$$= \dot{\theta} \hat{\underline{E}}_3 \times (\sin \theta \hat{\underline{E}}_2 + \cos \theta \hat{\underline{E}}_1)$$

(Note that
 $\left. \frac{d\hat{\underline{E}}_i}{dt} \right|_F = 0$)



Recognize that this equation can be expressed as

$$\left. \frac{d\hat{\underline{e}}_1}{dt} \right|_F = \underline{\omega}_{m|F} \times \hat{\underline{e}}_1, \text{ where } \underline{\omega}_{m|F} = \dot{\theta} \hat{\underline{E}}_3$$

Similarly, we can show that

$$\left. \frac{d\hat{\underline{e}}_2}{dt} \right|_F = \underline{\omega}_{m|F} \times \hat{\underline{e}}_2, \text{ and } \left. \frac{d\hat{\underline{e}}_3}{dt} \right|_F = \underline{\omega}_{m|F} \times \hat{\underline{e}}_3$$

$$\therefore \boxed{\left. \frac{d\hat{\underline{e}}_i}{dt} \right|_F = \underline{\omega}_{m|F} \times \hat{\underline{e}}_i} \quad \text{--- (I)}$$

We can show that for any vector $\underline{r}$, whose orientation changes but the magnitude does not change, the time derivative w.r.t fixed frame 'F' is:

simple rotation

$$\boxed{\left. \frac{d\underline{r}}{dt} \right|_F = \underline{\omega}_{m|F} \times \underline{r}} \quad \text{--- (II)}$$

Proof: $\underline{r} = r_1 \hat{\underline{e}}_1 + r_2 \hat{\underline{e}}_2 + r_3 \hat{\underline{e}}_3$ where r_1, r_2, r_3 are constant values

$$\left. \frac{d\underline{r}}{dt} \right|_F = \cancel{\frac{dr_1}{dt}} \hat{\underline{e}}_1 + r_1 \left. \frac{d\hat{\underline{e}}_1}{dt} \right|_F + \cancel{\frac{dr_2}{dt}} \hat{\underline{e}}_2 + r_2 \left. \frac{d\hat{\underline{e}}_2}{dt} \right|_F + \cancel{\frac{dr_3}{dt}} \hat{\underline{e}}_3 + r_3 \left. \frac{d\hat{\underline{e}}_3}{dt} \right|_F$$

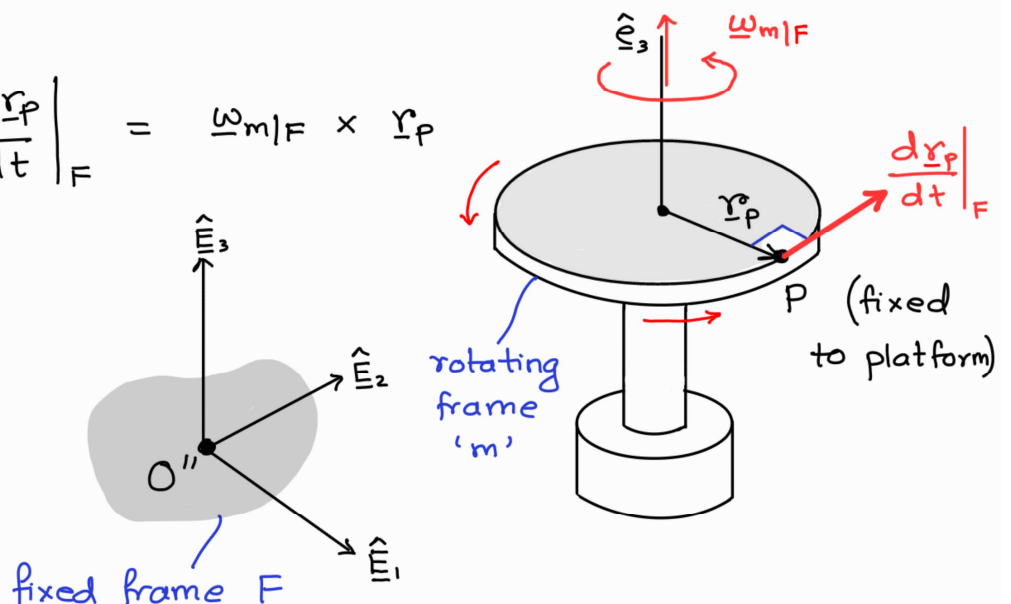
Use (I)

$$\begin{aligned} &= r_1 (\underline{\omega}_{m|F} \times \hat{\underline{e}}_1) + r_2 (\underline{\omega}_{m|F} \times \hat{\underline{e}}_2) + r_3 (\underline{\omega}_{m|F} \times \hat{\underline{e}}_3) \\ &= \underline{\omega}_{m|F} \times (r_1 \hat{\underline{e}}_1 + r_2 \hat{\underline{e}}_2 + r_3 \hat{\underline{e}}_3) \\ &= \underline{\omega}_{m|F} \times \underline{r} \end{aligned}$$

In regard to kinematics of a particle (or point), we have understood that derivative of the position vector of a pt P fixed to a moving frame 'm' that rotates relative to 'F' is given by

is given by

Velocity vector of P fixed in 'm' $= \left. \frac{d\underline{r}_P}{dt} \right|_F = \underline{\omega}_{m|F} \times \underline{r}_P$



Relationship of velocity and acceleration vectors of a point w.r.t rotating frame 'm' and fixed frame 'F'

We will establish the relationship for a general **time-varying** vector, say $\underline{A}(t)$, that is no longer fixed to the moving ref. frame 'm', but also changing w.r.t 'm'

$$\text{i.e. } \left. \frac{d\underline{A}}{dt} \right|_m \neq 0$$

$$\textcircled{1} \text{ If 'A' is a scalar } \Rightarrow \left. \frac{dA}{dt} \right|_F = \left. \frac{dA}{dt} \right|_m$$

(absolute $\Leftrightarrow$ ref. frame independent)

$$\textcircled{2} \text{ If } \underline{A} \text{ is a vector } \Rightarrow \left. \frac{d\underline{A}}{dt} \right|_F \neq \left. \frac{d\underline{A}}{dt} \right|_m$$

Let's see why!

$$\underline{A} = A_i \hat{E}_i \quad (\text{frame } F)$$

$$\underline{A} = a_i \hat{e}_i \quad (\text{frame } m)$$

} representation of the same vector $\underline{A}$ using two different sets of triads of unit vectors

$$\left. \frac{d\underline{A}}{dt} \right|_F = \left. \dot{\underline{A}} \right|_F = \left. \frac{d}{dt} (A_1 \hat{E}_1 + A_2 \hat{E}_2 + A_3 \hat{E}_3) \right|_F$$

$$= \left. \dot{A}_1 \hat{E}_1 + \dot{A}_2 \hat{E}_2 + \dot{A}_3 \hat{E}_3 \right|_F$$

$$\left. \frac{d\underline{A}}{dt} \right|_m = \left. \dot{\underline{A}} \right|_m = \left. \frac{d}{dt} (a_i \hat{e}_i) \right|_m = \left. \dot{a}_i \hat{e}_i \right|_m \leftarrow (\text{ESC})$$

$$\begin{aligned}\dot{\underline{A}}|_F &= \frac{d}{dt} (a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3) \Big|_F \\ &= \underbrace{\dot{a}_1 \hat{e}_1 + \dot{a}_2 \hat{e}_2 + \dot{a}_3 \hat{e}_3}_{\dot{\underline{A}}|_m} + \underbrace{a_1 \dot{\hat{e}}_1 + a_2 \dot{\hat{e}}_2 + a_3 \dot{\hat{e}}_3}_{\text{We can sub in the values of } \dot{\hat{e}}_i = \underline{\omega}_{m|F} \times \hat{e}_i \text{ from (I)}}\end{aligned}$$

$$a_1 \dot{\hat{e}}_1 + a_2 \dot{\hat{e}}_2 + a_3 \dot{\hat{e}}_3$$

$$= a_1 (\underline{\omega}_{m|F} \times \hat{e}_1) + a_2 (\underline{\omega}_{m|F} \times \hat{e}_2) + a_3 (\underline{\omega}_{m|F} \times \hat{e}_3)$$

$$= \underline{\omega}_{m|F} \times \underbrace{(a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3)}_{\underline{A}}$$

$$= \underline{\omega}_{m|F} \times \underline{A}$$

$$\dot{\underline{A}}|_F = \dot{\underline{A}}|_m + (\underline{\omega}_{m|F} \times \underline{A})$$

Even if the fixed frame 'F' was also a moving frame, the above equation would still hold.

Also, think that NO frame of reference is absolutely fixed!

e.g. Earth spins & revolves around the sun

If the components of vector $\underline{A}$ remain constant w.r.t. moving frame 'm'

$\dot{\underline{A}}|_m = 0$, then the rate of change of $\underline{A}$ is the cross-product of

$$\underline{\omega}_{m|F} \text{ and } \underline{A} \Rightarrow \dot{\underline{A}}|_F = \underline{\omega}_{m|F} \times \underline{A} \text{ (same as (II))}$$

Note: $\underline{\dot{A}}|_F \neq \underline{\dot{A}}|_m$, unless $\underline{\omega}_{m|F} = 0$ (or)

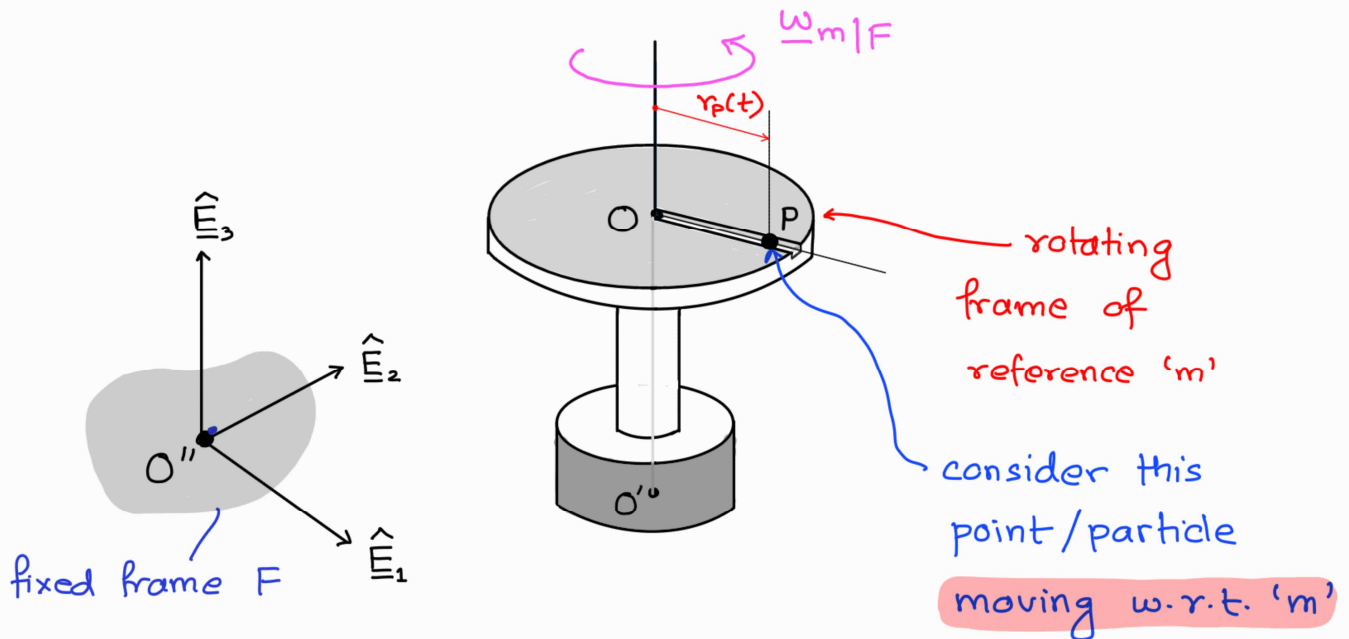
$\underline{\omega}_{m|F}$ is parallel to $\underline{A}$

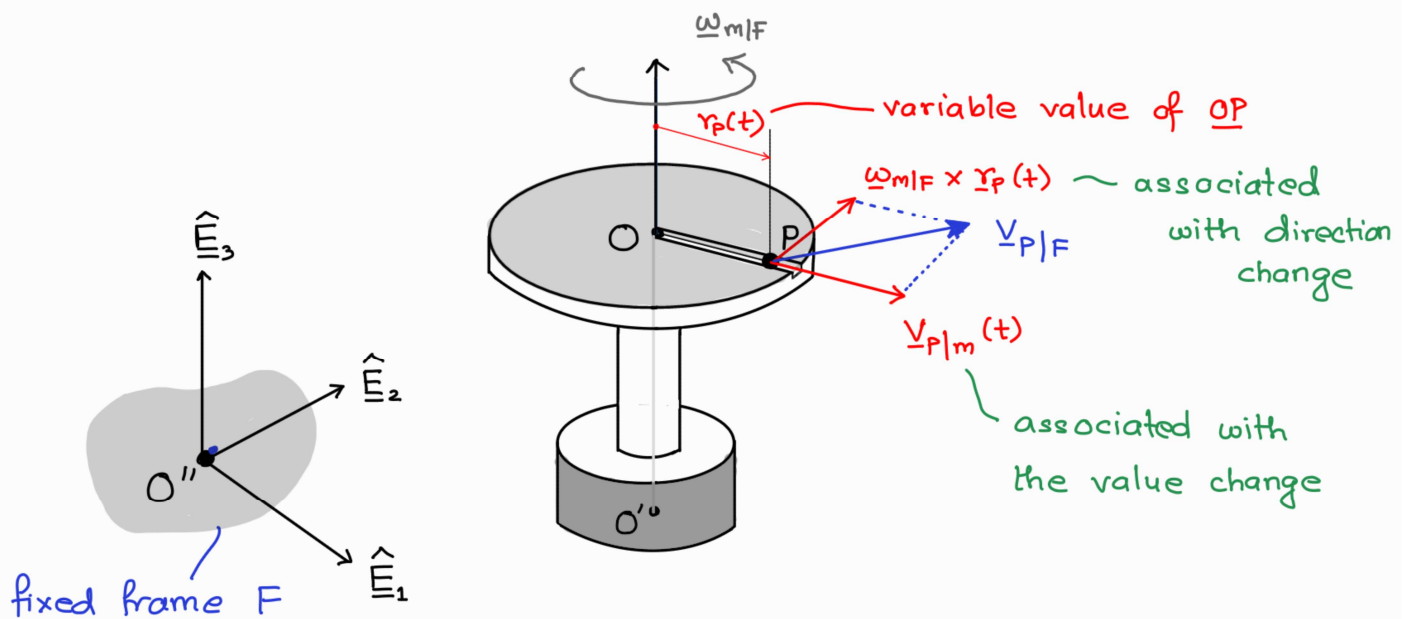
Relation of velocity vector of P wrt 'm' and 'F'

Substituting $\underline{A} = \underline{r}_P$, we get

$$\underline{v}_{P|F} = \left. \frac{d\underline{r}_P}{dt} \right|_F = \underline{v}_{P|m} + \underline{\omega}_{m|F} \times \underline{r}_P$$

To consider a physical example, let's now consider that pt P is no longer fixed to 'm' but free to move. Thus, $\underline{r}_P(t)$ now not only changes direction but may also change its magnitude.





Angular acceleration vector of rotating frame 'm' wrt 'F'

Angular acceleration is defined as the time derivative of the angular velocity, usually denoted by $\underline{\alpha} = \dot{\underline{\omega}}$

The angular velocity of a rotating frame 'm' is a vector which can be expressed in terms of the csys $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$ of the frame 'm'

$$\underline{\omega}_{m/F} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3$$

You will notice that, even if the rotation rates $\omega_1, \omega_2, \omega_3$ remain constant, there will be acceleration since the coordinate axes $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$ change with time for frame 'm'

To find the angular acceleration, we make use of the generic result $\dot{\underline{A}}|_F = \dot{\underline{A}}|_m + \underline{\omega}_{m/F} \times \underline{A}$ and substitute $\underline{A} = \underline{\omega}_{m/F}$

Diff w.r.t time

Angular
acceleration

$$\begin{aligned} \hookrightarrow \underline{\dot{a}}|_F &= \left(\underline{\dot{\omega}}_{m|F} \right)|_F = \left(\underline{\dot{\omega}}_{m|F} \right)|_m + \left(\underline{\omega}_{m|F} \times \underline{\omega}_{m|F} \right) \\ &= \underline{\dot{\omega}}_{m|F}|_m \end{aligned}$$

$\nearrow$
// to
each other

$\therefore \underline{\dot{a}}|_F$ (or $\underline{\dot{\omega}}_{m|F}$) is the same in 'F' or 'm'

We can also use analytical derivative of vectors to derive it.

The angular acceleration of the moving frame 'm' in terms of the csys of moving frame can be expressed as:

Using ESC

$$\underline{\dot{\omega}}_{m|F} = \frac{d}{dt} \left(\omega_i \hat{e}_i \right)|_F = \dot{\omega}_i \hat{e}_i|_F + \omega_i \dot{\hat{e}}_i|_F = \dot{\omega}_i \hat{e}_i + \omega_i (\underline{\omega}_{m|F} \times \hat{e}_i)$$

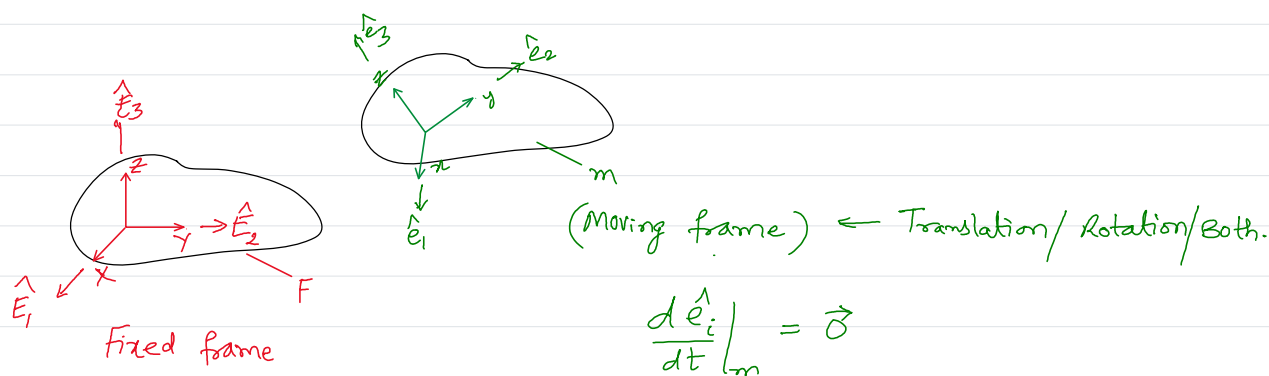
vector
scalar

$$\underline{\dot{\omega}}_{m|F} = \underbrace{\dot{\omega}_1 \hat{e}_1 + \dot{\omega}_2 \hat{e}_2 + \dot{\omega}_3 \hat{e}_3}_{\substack{\Downarrow \\ \text{due to change of} \\ \text{rotation rates w.r.t 'm'}}} + \underbrace{\omega_1 (\underline{\omega}_{m|F} \times \hat{e}_1) + \omega_2 (\underline{\omega}_{m|F} \times \hat{e}_2) + \omega_3 (\underline{\omega}_{m|F} \times \hat{e}_3)}_{\substack{\Downarrow \\ \text{due to change of orientation} \\ \text{of csys of 'm' w.r.t 'F'}}}$$

$$= \underline{\dot{\omega}}_{m|F}|_m + \underline{\omega}_{m|F} \times (\omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3)$$

$$= \underline{\dot{\omega}}_{m|F}|_m + \underline{\omega}_{m|F} \times \underline{\omega}_{m|F}$$

$$= \underline{\dot{\omega}}_{m|F}|_m$$



$$\left. \frac{d\hat{E}_i}{dt} \right|_F = \vec{0}$$

But, generally, $\left. \frac{d\hat{e}_i}{dt} \right|_F \neq 0$

$$\left. \frac{d\hat{e}_1}{dt} \right|_F = a_{11}\hat{e}_1 + a_{12}\hat{e}_2 + a_{13}\hat{e}_3 = \dot{\hat{e}}_1|_F$$

$$\left. \frac{d\hat{e}_2}{dt} \right|_F = a_{21}\hat{e}_1 + a_{22}\hat{e}_2 + a_{23}\hat{e}_3 = \dot{\hat{e}}_2|_F$$

and $\left. \frac{d\hat{e}_3}{dt} \right|_F = a_{31}\hat{e}_1 + a_{32}\hat{e}_2 + a_{33}\hat{e}_3 = \dot{\hat{e}}_3|_F$

$$\left. \frac{d\hat{e}_i}{dt} \right|_F = a_{ij}\dot{\hat{e}}_j = \dot{\hat{e}}_i|_F$$

Now, $\left(\hat{e}_1 \cdot \hat{e}_1 \right) \Big|_F = 1$

Differentiate
w.r.t
time
on both
sides

$$\Rightarrow \left(\hat{e}_1 \cdot \frac{d\hat{e}_1}{dt} \right) \Big|_F = 0 \Rightarrow a_{11} = 0$$

similarly, $\left(\hat{e}_2 \cdot \hat{e}_2 \right) \Big|_F = 1 \Rightarrow a_{22} = 0$

and $\left(\hat{e}_3 \cdot \hat{e}_3 \right) \Big|_F = 1 \Rightarrow a_{33} = 0$

$$\text{Also, } \left(\hat{e}_1 \cdot \hat{e}_2 \right) \Big|_F = 0$$

$$\Rightarrow \left(\frac{d\hat{e}_1}{dt} \cdot \hat{e}_2 + \hat{e}_1 \cdot \frac{d\hat{e}_2}{dt} \right) \Big|_F = 0$$

$$\Rightarrow a_{12} + a_{21} = 0$$

$$\Rightarrow a_{12} = -a_{21}$$

$$\text{Similarly } a_{23} = -a_{32}$$

$$a_{13} = -a_{31}$$

$$\begin{bmatrix} \dot{\hat{e}}_1 \\ \dot{\hat{e}}_2 \\ \dot{\hat{e}}_3 \end{bmatrix} \Big|_F = \begin{bmatrix} \overset{0}{\cancel{a_{11}}} & a_{12} & a_{13} \\ \overset{0}{\cancel{a_{21}}} & \overset{0}{\cancel{a_{22}}} & a_{23} \\ a_{31} & \overset{0}{\cancel{a_{32}}} & \overset{0}{\cancel{a_{33}}} \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$$

↑
Skew Symmetric matrix $\Leftrightarrow a_{ij} = -a_{ji}$

$$\dot{\hat{e}}_1 \Big|_F = 0 \hat{e}_1 + a_{12} \hat{e}_2 - a_{31} \hat{e}_3 = (a_{12} \hat{e}_3 + a_{23} \hat{e}_1 + a_{31} \hat{e}_2) \times \hat{e}_1$$

$$\dot{\hat{e}}_2 \Big|_F = -a_{12} \hat{e}_1 + 0 \hat{e}_2 + a_{23} \hat{e}_3 = (a_{12} \hat{e}_3 + a_{23} \hat{e}_1 + a_{31} \hat{e}_2) \times \hat{e}_2$$

$$\dot{\hat{e}}_3 \Big|_F = a_{31} \hat{e}_1 - a_{23} \hat{e}_2 + 0 \hat{e}_3 = (a_{12} \hat{e}_3 + a_{23} \hat{e}_1 + a_{31} \hat{e}_2) \times \hat{e}_3$$

$$\boxed{\vec{\omega}_{m/F} = a_{12} \hat{e}_3 + a_{23} \hat{e}_1 + a_{31} \hat{e}_2}$$

Angular velocity of frame 'm' w.r.t frame 'F'

$$\dot{\hat{e}}_i \Big|_F = \vec{\omega}_{m/F} \times \hat{e}_i$$

$\Rightarrow \dot{\hat{e}}_i \Big|_F = \vec{0}$ when $\vec{\omega}_{m/F} = \vec{0}$ (i.e., 'm' is undergoing PURE TRANSLATION w.r.t 'F')

Recall:

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

$$\begin{aligned} \hat{e}_1 \times \dot{\hat{e}}_1 \Big|_F &= \hat{e}_1 \times (\vec{\omega}_{m/F} \times \hat{e}_1) \\ &= (\hat{e}_1 \cdot \hat{e}_1) \vec{\omega}_{m/F} - \hat{e}_1 (\vec{\omega}_{m/F} \cdot \hat{e}_1) \end{aligned}$$

$$\Rightarrow \hat{e}_1 \times \dot{\hat{e}}_1 \Big|_F = \vec{\omega}_{m/F} - a_{23} \hat{e}_1$$

Similarly, $\hat{e}_2 \times \dot{\hat{e}}_2 \Big|_F = \vec{\omega}_{m/F} - a_{31} \hat{e}_2$

and $\hat{e}_3 \times \dot{\hat{e}}_3 \Big|_F = \vec{\omega}_{m/F} - a_{12} \hat{e}_3$

$$\left[(\hat{e}_1 \times \dot{\hat{e}}_1) + (\hat{e}_2 \times \dot{\hat{e}}_2) + (\hat{e}_3 \times \dot{\hat{e}}_3) \right] \Big|_F = 3 \vec{\omega}_{m/F} - \underbrace{(a_{23} \hat{e}_1 + a_{31} \hat{e}_2 + a_{12} \hat{e}_3)}_{\vec{\omega}_{m/F}}$$

$$= 2 \vec{\omega}_{m/F}$$

$$\Rightarrow \vec{\omega}_{m/F} = \frac{1}{2} \left[(\hat{e}_1 \times \dot{\hat{e}}_1) + (\hat{e}_2 \times \dot{\hat{e}}_2) + (\hat{e}_3 \times \dot{\hat{e}}_3) \right]$$

$$\boxed{\vec{\omega}_{m/F} = \frac{1}{2} [\hat{e}_i \times \dot{\hat{e}}_i]}$$