PVW and Conservative forces

Recap of Conservative forces

Recall that when a force field does work that depends only upon the initial and final positions of the force, and it is independent of the path it travels, then the force is called conservative force

e.g. weight of a body
force of a linear spring
} examples of conservative forces

Also recall that conservative forces are associated with a scalar potential function V, such that the conservative force field is related to the potential function V as

$$\underline{F}(\underline{r}) = -\frac{d}{d\underline{r}} V(\underline{r})$$

conservative force field

potential function

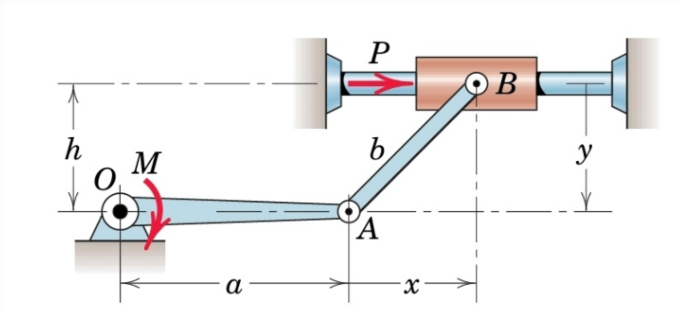
Work done by conservative forces can be expressed in terms of the potential function V

$$\Rightarrow \quad \underbrace{dW}_{\text{Positive work done by conservative force}} = \underline{F} \cdot d\underline{r} = \underbrace{-\,dV}_{\text{Decrease in potential energy}}$$

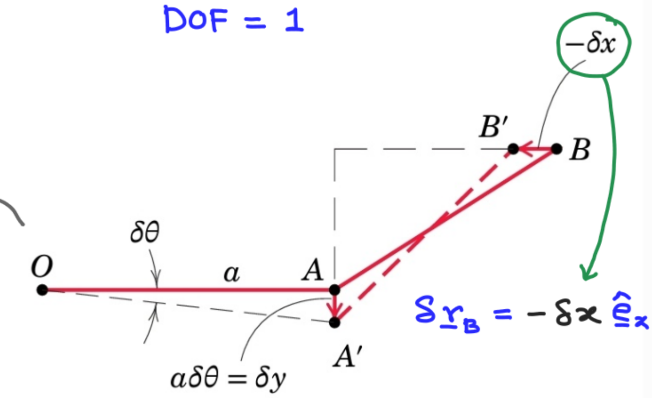
Recall the principle of virtual work (PVW) for rigid bodies

The virtual work, δW, done by applied external forces $\underline{F}_1, \ldots, \underline{F}_N$ and resultant couple $\underline{C}$ in moving through virtual displacements, $\delta \underline{r}_1, \ldots, \delta \underline{r}_N$, and $\delta \underline{\theta}$, respectively, due to virtual displacements at the DOFs of the system, one at a time, is zero if the body is in static equilibrium

$$\delta W = \sum_{i=1}^{N} \underline{F}_i \cdot \delta \underline{r}_i + \underline{C} \cdot \delta \underline{\theta} = 0$$



DOF = 1



If the system is in static equilibrium:

$$\delta W = (P \hat{\underline{e}}_x) \cdot (-\delta x \, \hat{\underline{e}}_x) + M \delta \theta = 0$$

$$\Rightarrow \quad -P \delta x + M \delta \theta = 0$$

PVW for conservative forces

Suppose we have a system in static equilibrium and all external forces that can do non-zero virtual work are known to be conservative, then we can write the statement of virtual work in terms of potential energy

$$\delta W = \sum_{i=1}^{N} \underbrace{\left(F_i^{cons} \cdot \delta \underline{r}_i \right)}_{-\delta V_i}$$

δV_i → variation in potential energy for the ith conservative force due to virtual displacement $\delta \underline{r}_i$

e.g.

$$\delta V_i = \frac{dV_i}{dq} \delta q$$

(for a virtual displacement at a DOF q of a system)

Thus, if only 'N' conservative forces do non-zero virtual work then the total virtual work done may be written as

small but arbitrary and $\neq 0$

$$\delta W = -\sum_{i=1}^{N} \frac{dV_i}{dq} \delta q = 0$$

$$\Rightarrow \sum_{i=1}^{N} \frac{dV_i}{dq} = 0$$

$$\Rightarrow \frac{d}{dq} \left\{ \sum_{i=1}^{N} V_i \right\} = 0$$

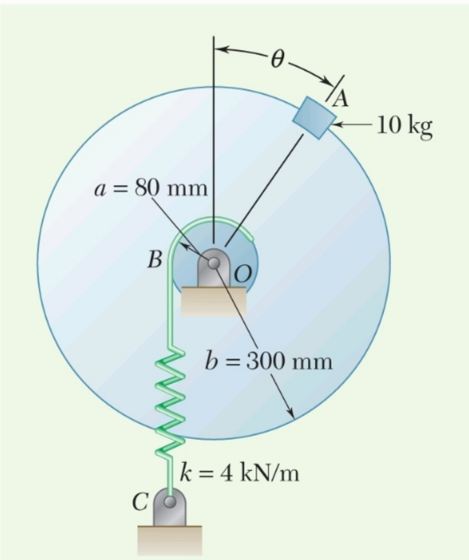
Let $V = \sum_{i=1}^{N} V_i$ be the total potential energy due to all N conservative forces

then : $$\boxed{\frac{dV}{dq} = 0}$$

If the system is in static equilibrium, the derivative of its potential energy w.r.t. the DOF is zero

If there are several DOFs, the partial derivatives of V w.r.t each DOF must be zero for the system to be in equilibrium.

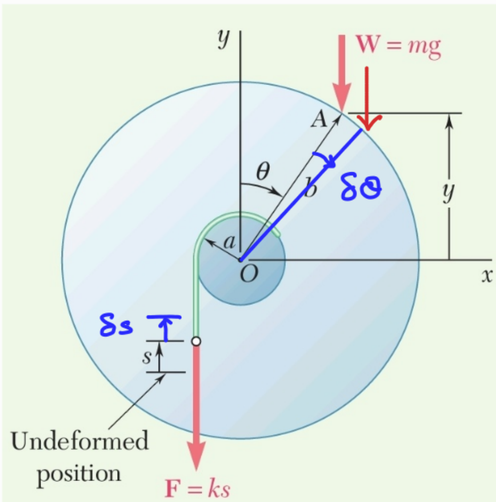
Example:



A 10-kg block is attached to the rim of a 300-mm-radius disk as shown. Knowing that spring BC is unstretched when $\theta = 0$, determine the position or positions of equilibrium, using PVW.

Solu :

1) # of DOFs $= 1 \quad (q \equiv \theta)$



2) Draw FBD of the virtually displaced configuration

$\delta s \rightarrow$ virtual deflection of spring

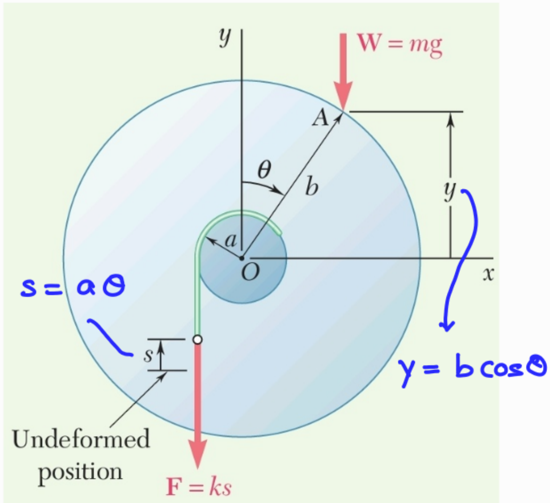
$\delta \theta \rightarrow$ " rotation of the DOF

3) Identify the forces that do non-zero virtual work

$$\left.\begin{aligned} \underline{W} &= -mg\,\hat{e}_y \\ \underline{F} &= -ks\,\hat{e}_y \end{aligned}\right\} \text{conservative forces} \Rightarrow \text{can use } \frac{dV}{dq} = 0$$

Weight of the disk does no virtual work

4) Choose a coordinate system and determine total potential energy in terms of virtual disp at DOF 'q'



Spring : $V_s = \frac{1}{2} k s^2$

Block : $V_b = mgy$

Total PE, $V(\theta) = V_s + V_b$

$$= \frac{1}{2} k s^2 + mgy$$

$$= \frac{1}{2} k a^2 \theta^2 + mg\, b\cos\theta$$

5) For static equilibrium, set $\dfrac{dV(q)}{dq} = 0$

$$\frac{dV(\theta)}{d\theta} = 0 \Rightarrow ka^2\theta - mg\sin\theta = 0$$

$$\Rightarrow \sin\theta = \frac{ka^2}{mgb}\,\theta$$

Solve by trial & error for θ $\Rightarrow$ $\theta = 0.902$ rad

Stability of Equilibrium

The previous equation $\boxed{\frac{dV}{dq} = 0}$ tells the requirement that the equilibrium configuration of a mechanical system is one for which the total potential energy V of the system has a stationary value.

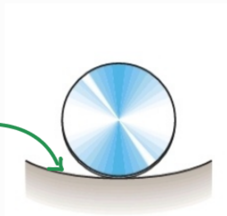
$$\boxed{\frac{dV}{dq} = 0} \Leftrightarrow \text{Equilibrium}$$

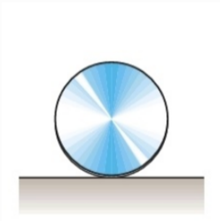
minimum (stable)

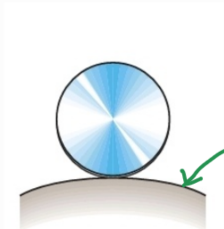
constant (neutral)

maximum (unstable)

concave up







convex up

$$\frac{d^2V}{dq^2} > 0$$

$$\frac{d^2V}{dq^2} < 0$$