Principle of Virtual Work (PVW)

The previous approach of analyzing a truss or frame under static equilibrium, they were analyzed by

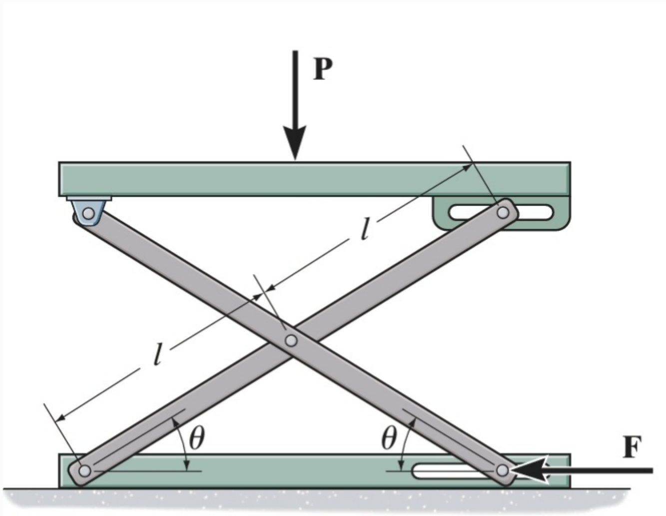
1) Isolating the bodies using FBDs

2) Applying conditions of static equilibrium

   - Sum of forces $\Sigma F_x = \Sigma F_y = 0$
   - Sum of moments $\Sigma M_* = 0$

Limitations of this approach

Not ideal for bodies composed of interconnected members, where the interconnected members can move relative to each other



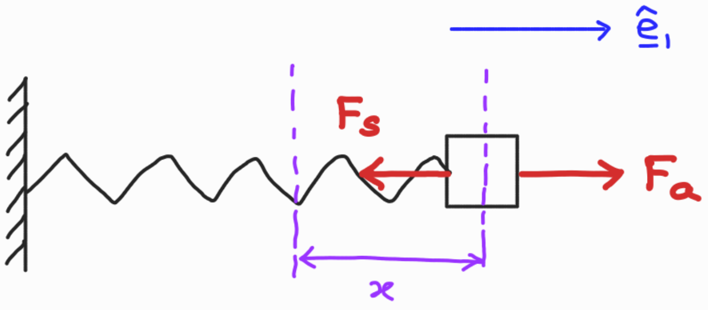
Say you want to hold the horizontal slider in equilibrium at a certain position with angle θ, what is the relation between F and P ?

For analyzing this body, you would need to draw FBDs for each RB, write the force and moment balance eqns for each RB, and then solve a system of equations to find the force F and relate it with P $\Rightarrow$ Tedious way

A faster and more efficient way of determining various possible equilibrium positions is through using PVW

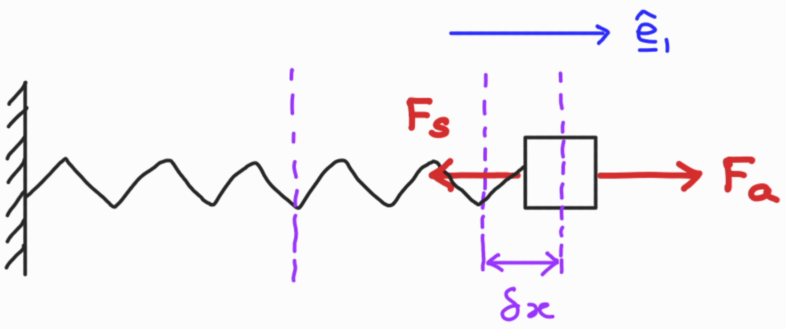
Definition of virtual work for a particle

Consider a particle with mass attached to a spring & pulled by an applied force F_a



When in equilibrium, the spring force $F_s = -kx\,\hat{e}_1$ (with 'x' being the stretch of spring from undeformed position) must be equal to applied force F_a

$$\overset{+}{\rightarrow} \sum F_x = 0 \Rightarrow F_s + F_a = 0$$
$$\Rightarrow -kx + F_a = 0$$



To introduce virtual work, imagine that the mass is infact not at its equilibrium position but at an (incorrect) non-equilibrium position $x + \delta x$.

This imaginary displacement δx is called virtual disp.

symbol 'δ' is called variation

Now, define virtual work δW done by a force to be the equilibrium force times this small imaginary disp δx

Note no real work has been performed since δx is not real displacement; this is more like a "thought experiment".

The virtual work done by the spring force, $\delta W_s = F_s \, \delta x$
$= -kx \, \delta x$

The virtual work done by the applied force, $\delta W_a = F_a \, \delta x$

$$\boxed{\begin{aligned} \text{Total virtual work} = \delta W &= \delta W_s + \delta W_a \\ &= (-kx + F_a)\,\delta x \end{aligned}}$$

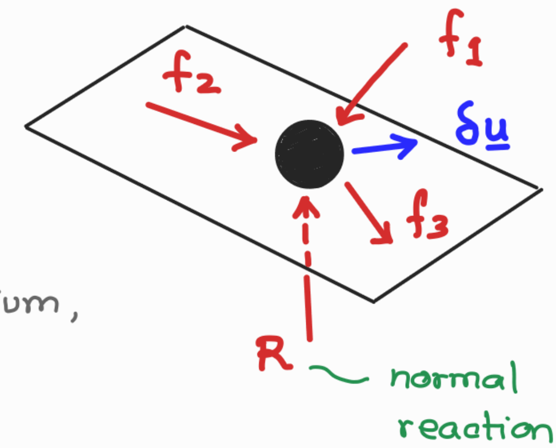
Two ways of viewing this expression

If the system is in equilibrium (i.e. $-kx + F_a = 0$), then virtual work is zero, $\delta W = 0$	If the virtual work is zero, then, since δx is arbitrary, the system must be in equilibrium $\Rightarrow -kx + F_a = 0$

Geometric (or Kinematic) constraints

In many practical problems, the particle will usually be constrained to move in only certain directions.

For example, the ball (treated as a particle) rolling on a table is not allowed to go "through" the table. If the ball is in static equilibrium, then



$$\underbrace{\underline{R}}_{\text{reaction force}} + \underbrace{\sum \underline{f}_i}_{\text{applied forces}} = \underline{0}$$

If a virtual displacement $\delta \underline{u}$ is given such that it is geometrically admissible (i.e. the constraints are not violated) then $\delta \underline{u}$ and $\underline{R}$ must be perpendicular, that is the virtual work done by the reaction force is zero, and

$$\delta W = \left(\sum \underline{f}_i \right) \cdot \delta \underline{u} = 0$$

This is one of the benefits of virtual work → one does not need to calculate the forces of constraint $\underline{R}$ in order to determine the forces $\underline{f}_i$ which maintain the particle in equilibrium

The term kinematically admissible (or geometrically admissible) is used to mean one that does not violate the constraints and one arrives at the version of principle of virtual work which is often used in practice:

PVW (for particle): A particle is in static equilibrium under the action of a system of forces if the total work done by the forces (excluding workless forces) is zero for any kinematically admissible virtual displacement of the particle

Principle of virtual work for rigid bodies

We can easily extend the PVW for a single particle to an RB (treated as system of many particles whose mutual distances are fixed). Because the virtual work done on each particle of the RB in equilibrium is zero, it follows that the virtual work done on the entire RB is zero.

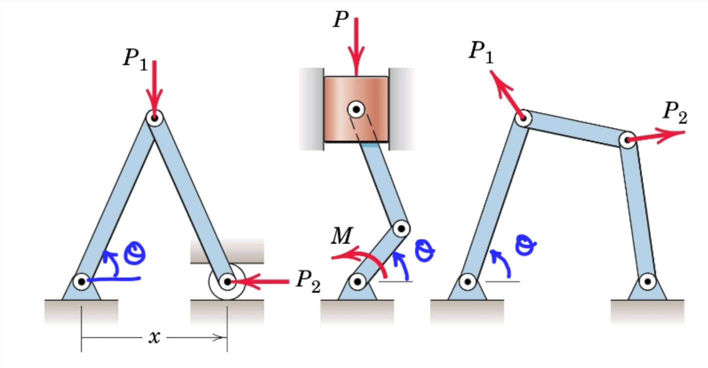
But before that it is important to introduce the concept of degrees of freedom of an RB (or a system of RBs)

Degrees of Freedom (DOFs) of a system

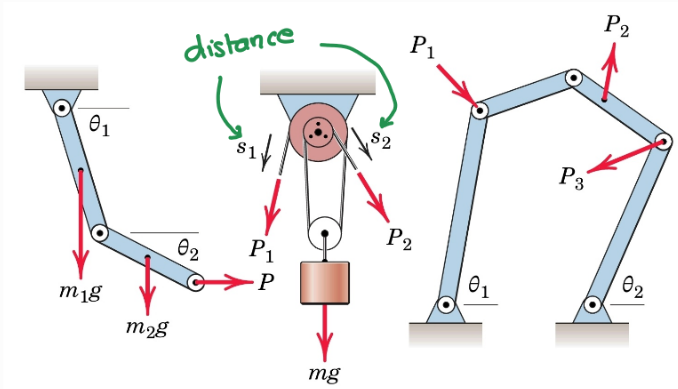
To write out the statement of PVW for RBs, we introduce the idea of degrees of freedom of an RB (or a system of RBs) as the number of independent coordinates needed to specify completely the configuration of the system

Angles

Distances



1 DOF systems



2 DOF systems

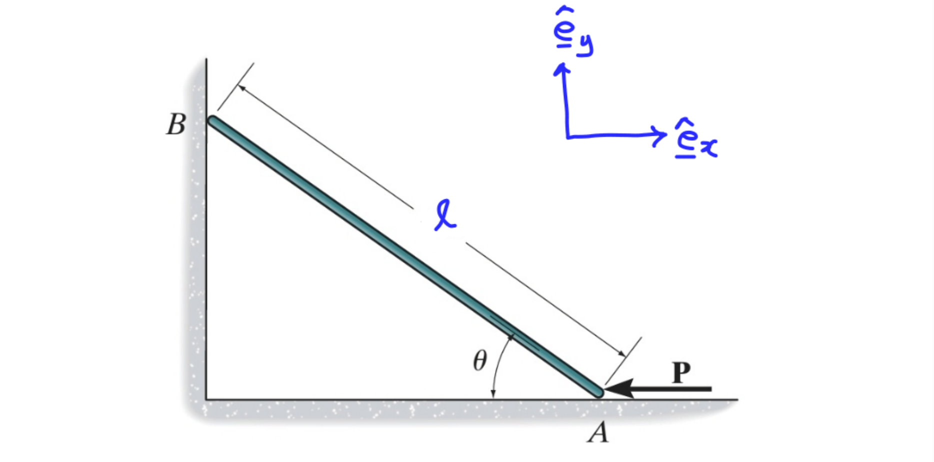
For defining virtual work for RBs (and system of RBs), we use variations of displacements and rotations along DOFs one by one, and set the virtual work done by applied external forces $\underline{F}_1, \dots, \underline{F}_N$ and resultant couple $\underline{C}$ as zero to get the relation between forces at static equilibrium

$$\delta W = \sum_{i=1}^{N} \underline{F}_i \cdot \delta \underline{r}_i + \underline{C} \cdot \delta \underline{\theta} = 0$$

virtual displacement (translation) along the line of action of $\underline{F}_i$

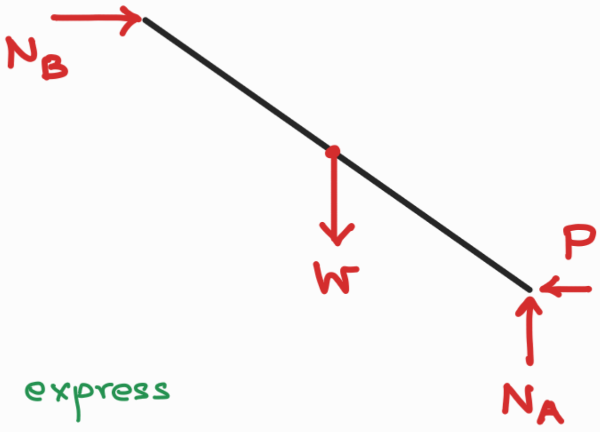
Let's consider an example of using PVW for an RB

F11–2. Determine the magnitude of force **P** required to hold the W-kg smooth rod in equilibrium at θ



Traditional method

Draw FBD



$\xrightarrow{+} \sum F_x = 0 \quad \Rightarrow \quad N_B = P$

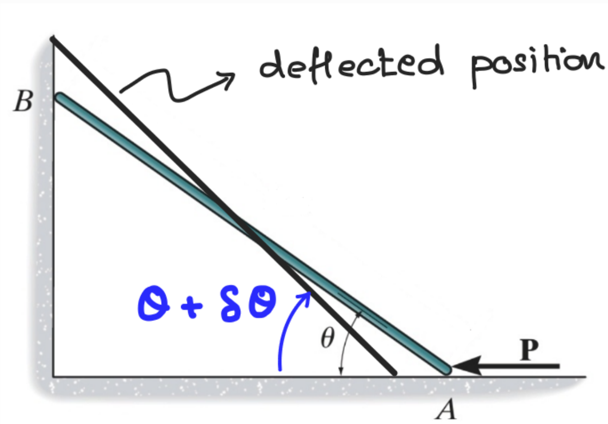
try to express in terms of W

$+\uparrow \sum F_y = 0 \quad \Rightarrow \quad N_A = W$

$\circlearrowleft + \sum M_A = 0 \quad \Rightarrow \quad -\cancel{N_B}^{\,P}(\ell \sin\theta) + W\left(\frac{\ell}{2}\cos\theta\right) = 0$

$\Rightarrow \quad P = \frac{W}{2}\cot\theta$

PVW method



$\delta\theta \rightarrow$ virtual displacement at the DOF

1) Identify the DOF : Has 1 dof $\rightarrow \theta$

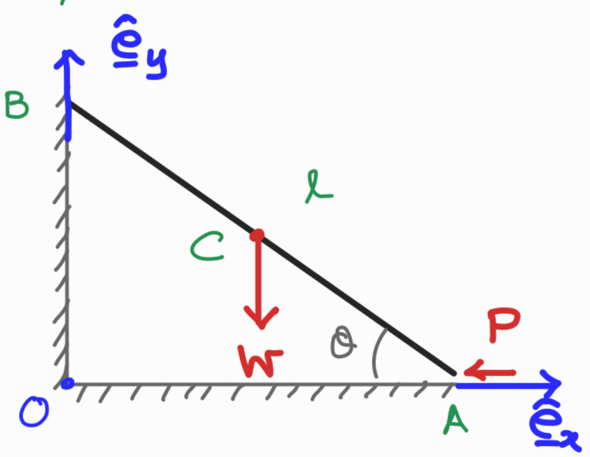
2) Draw the deflected position of the system when the system undergoes a virtual displacemen δq

3) Identify the forces that do non-zero virtual work (i.e. non-reaction forces/couples)

In this case, the self-weight W and horizontal force P do non-zero virtual work.

4) Identify the virtual displacements $\delta \underline{r}_i$ to find δW.

To find $\delta \underline{r}_i$, choose a csys s.t. its origin and the coordinate axes do not undergo any virtual displacement



$$\underline{r}_A = \ell \cos\theta \; \hat{\underline{e}}_x$$

$$\underline{r}_C = \frac{\ell}{2}\cos\theta \; \hat{\underline{e}}_x + \frac{\ell}{2}\sin\theta \; \hat{\underline{e}}_y$$

$$\delta \underline{r}_A = -\ell \sin\theta \; \delta\theta \; \hat{\underline{e}}_x$$

$$\delta \underline{r}_C = -\frac{\ell}{2}\sin\theta \; \delta\theta \; \hat{\underline{e}}_x + \frac{\ell}{2}\cos\theta \; \delta\theta \; \hat{\underline{e}}_y$$

5) Express the virtual work of each force and couple in the PVW equation in terms of δq (here $\delta\theta$)

$$\underline{P} = -P\,\hat{\underline{e}}_x \,, \qquad \underline{W} = -W\,\hat{\underline{e}}_y$$

not present

$$\delta W = \sum \underline{F}_i \cdot \delta \underline{r}_i + \cancel{\underline{C}} \cdot \delta\underline{\theta} = 0$$

$$= \underline{P} \cdot \delta \underline{r}_A + \underline{W} \cdot \delta \underline{r}_C$$

$$= (-P\,\hat{\underline{e}}_x) \cdot (-\ell \sin\theta \; \delta\theta \; \hat{\underline{e}}_x)$$

$$+ (-W\,\hat{\underline{e}}_y) \cdot \left(-\frac{\ell}{2}\sin\theta \; \delta\theta \; \hat{\underline{e}}_x + \frac{\ell}{2}\cos\theta \; \delta\theta \; \hat{\underline{e}}_y\right)$$

$$= \left(P\ell \sin\theta - \frac{W\ell}{2}\cos\theta \right)\delta\theta = 0$$

6) Factor out the common displacement from all the terms, and solve for the unknown force or couple.

Since $\delta\theta$ is arbitrary :

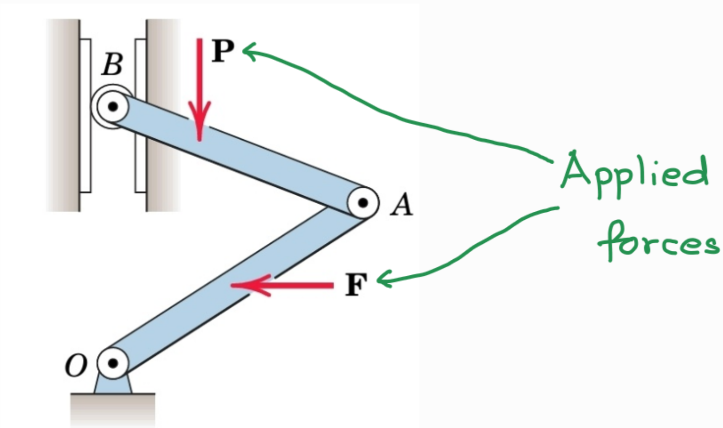
$$P\cancel{\ell}\sin\theta - \frac{W\cancel{\ell}}{2}\cos\theta = 0 \qquad (\ell \neq 0)$$

$$\Rightarrow \boxed{P = \frac{W}{2}\cot\theta} \qquad \text{(same as before)}$$

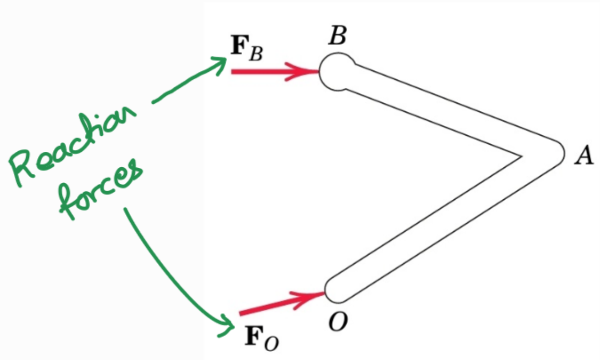
In PVW, several unknowns (N_A, N_B) drop out naturally

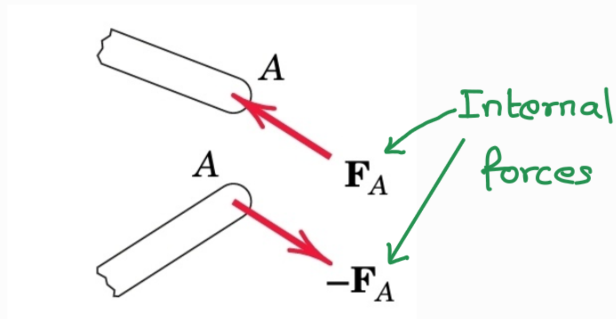
For a system of interconnected RBs, there arises 3 types of forces

a) Externally applied forces capable of doing virtual work during virtual displacements



b) Reaction forces which act at supports, and does no virtual work





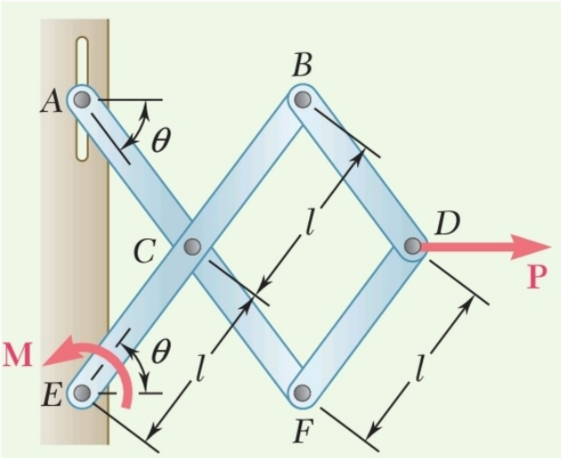
c) Internal forces are forces in the connections between members. During any possible movement of the system or its parts, the net work done by the internal forces at the connections is zero. This is because internal forces always exist in pairs of equal & opposite forces

and hence PVW for multiple interconnected RBs remain the same as stated for a single RB!

Ex

Using the method of virtual work, determine the magnitude of the couple **M** required to maintain the equilibrium of the mechanism shown.

Assume members are weightless

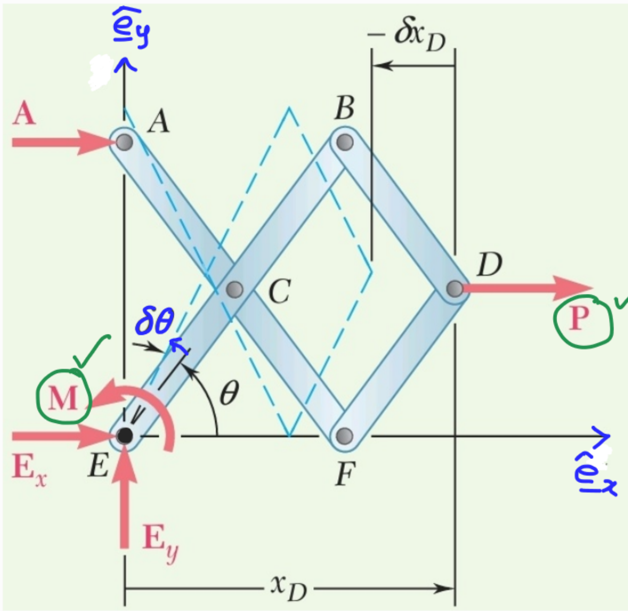


1) Identify DOF $\rightarrow$ θ (DOF = 1)

2) Draw the deflected config. by inducing virtual displacement

$\theta \rightarrow \theta + \delta\theta$

3) Identify forces that do non-zero virtual work



Force : $\underline{P} = P\,\hat{\underline{e}}_x$

Couple : $\underline{M} = M\,\hat{\underline{e}}_z$

Forces A_x, E_x, E_y, and the internal forces are workless

4) Choose a csys and determine $\delta \underline{r}_D$

Choose origin at E with the csys shown in figure

$$\underline{r}_D = x_D\,\hat{\underline{e}}_x + y_D\,\hat{\underline{e}}_y$$

$$= 3l\cos\theta\,\hat{\underline{e}}_x + l\sin\theta\,\hat{\underline{e}}_y$$

$$\delta\underline{r}_D = \left(-3l\sin\theta\,\hat{\underline{e}}_x + l\cos\theta\,\hat{\underline{e}}_y\right)\delta\theta$$

5) Express the virtual work of each force and couple in the PVW equation in terms of δq (here $\delta\theta$)

$$\delta W = \underline{P} \cdot \delta \underline{r}_D + M\,\delta\theta$$

$$= (P\,\hat{\underline{e}}_x) \cdot (-3\ell \sin\theta\,\hat{\underline{e}}_x + \ell\cos\theta\,\hat{\underline{e}}_y)\,\delta\theta + M\,\delta\theta$$

$$= (-3P\ell\sin\theta + M)\,\delta\theta$$

6) Factor out the common displacement from all the terms, and solve for the unknown force or couple.

$$\delta W = 0$$

$$\Rightarrow -3P\ell\sin\theta + M = 0 \qquad [\because \delta\theta \text{ is arbitrary}]$$

$$\Rightarrow \boxed{M = 3P\ell\sin\theta}$$