Impact of rigid bodies

Impact means collison of two bodies in which large forces act for a very short duration leading to finite appreciable impulses.

e.g. impact of hammer on a pile

impact of meteorite on earth

In general, impact phenomena is complex and the modeller usually makes simplifications:

Simplifications:

1> Rigid body assumption just before & after impact

2> Almost zero contact time (impulses due to finite forces over the duration of contact is zero)

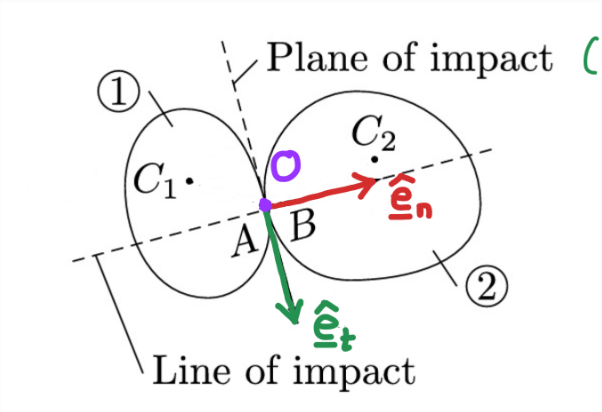
3> Impulsive forces (arising due contact) and the energy dissipation is modelled in terms of an empirical parameter 'e' ← coefficient of restitution

$e = 0$ (Plastic collision) $e = 1$ (Elastic collision)

$0 < e < 1$ (Inelastic collision)

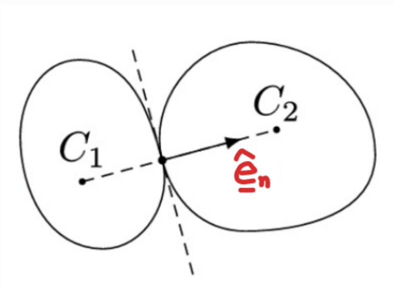
4) The impulsive forces cause instantaneous change in velocities of the bodies, without changing their positions

Consider impact of two unconstrained RBs, so that point A of body ① of mass m_1 collides with point B of body ② of mass m_2



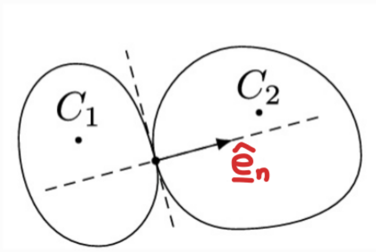
(tangent plane at the contact pt O which is coincident with A and B)

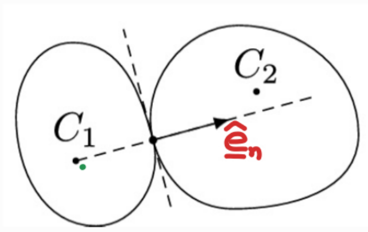
Central Impact



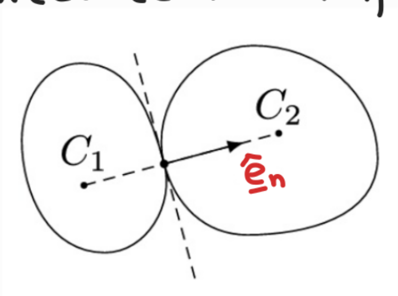
The COMs of the two bodies lie on the line of impact

Non-central Impact

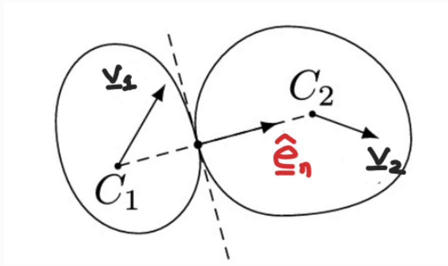




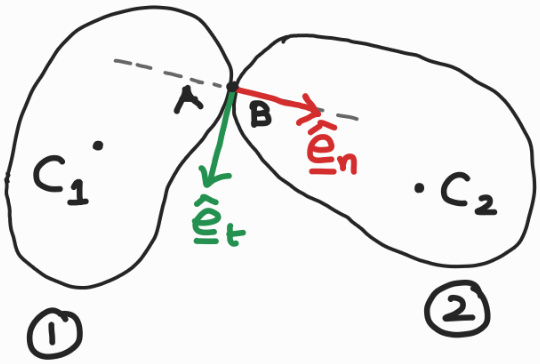
Direct central impact



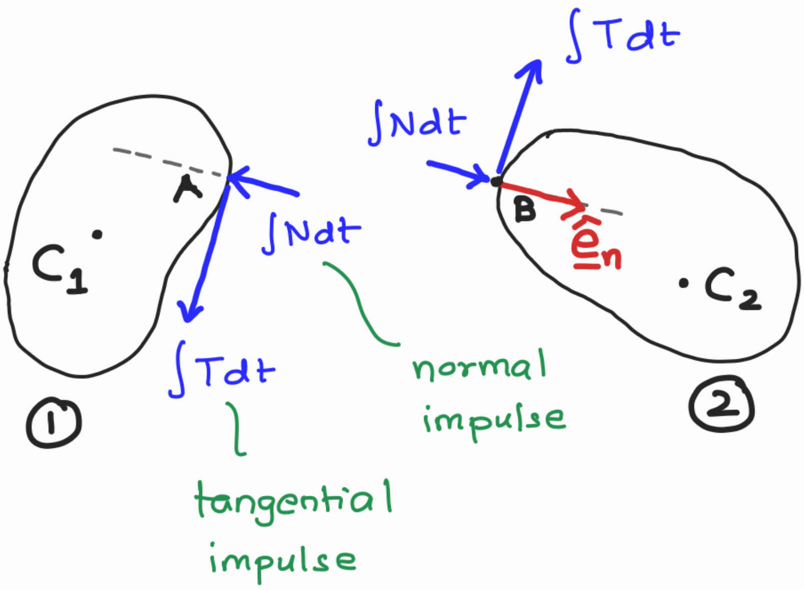
Oblique central impact



Smooth collision vs Non-smooth collision



FBD



Normal component of impulse : $\int N \, dt$

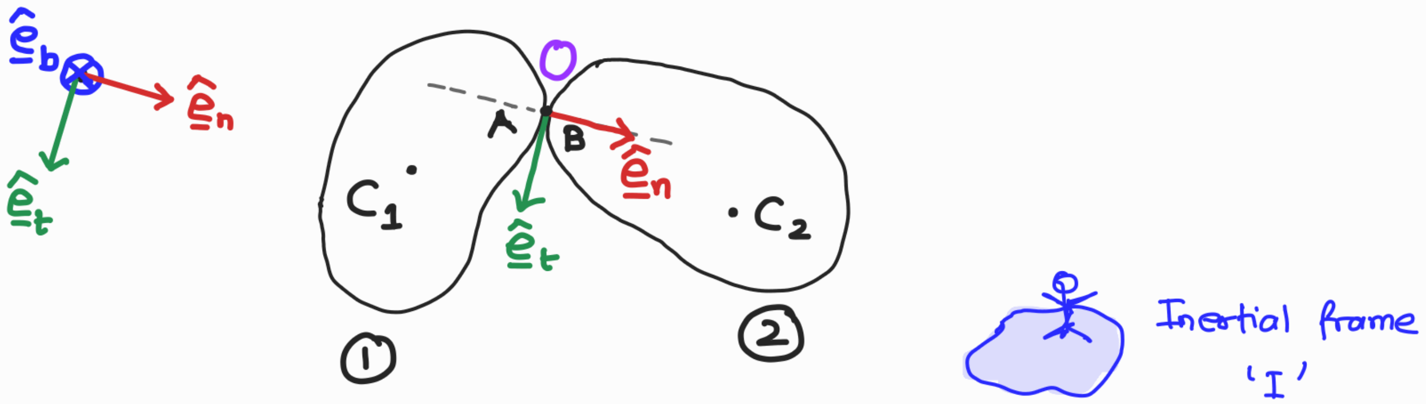
Tangential component of impulse : $\int T \, dt$ ← may arise due to friction

If bodies are "smooth", then $\int T \, dt = 0$

Otherwise $\int T \, dt \neq 0$

In this course, bodies are assumed to be smooth and we will look at smooth impact / collision problems

Impact / Collision Problem Setup:



1) Coincident pt 'O' is a geometric point that is not attached to any of the RBs but is fixed in the inertial frame $\Rightarrow$ acc of pt 'O' is zero, i.e., $\underline{a}_{O|I} = \underline{0}$

2) Points A ($\in$ RB 1) and B ($\in$ RB 2) do not move during the very short duration of collision

→ although $\underline{v}_A$ and $\underline{v}_B$ are not zero but finite!

→ acceleration of pts A and B are also not zero, however, acceleration of the geometric point 'O' is zero w.r.t the inertial frame

Before impact

RB ① : $\underline{v}_{c_1}$, $\underline{\omega}_1$

RB ② : $\underline{v}_{c_2}$, $\underline{\omega}_2$

known

After impact

RB ① : $\underline{v}_{c_1}'$, $\underline{\omega}_1'$

RB ② : $\underline{v}_{c_2}'$, $\underline{\omega}_2'$

to be found

3 components each $\Rightarrow$ 12 unknowns

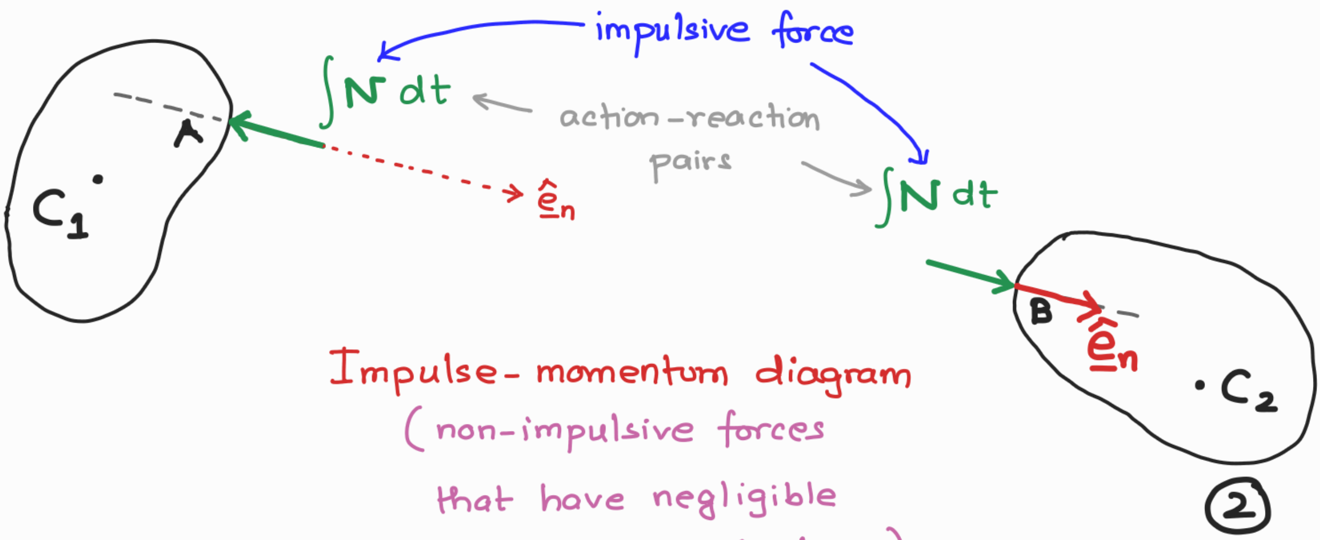
$$\underline{v}_{c_1} = v_{t_1}\underline{\hat{e}}_t + v_{n_1}\underline{\hat{e}}_n + v_{b_1}\underline{\hat{e}}_b$$

$$\underline{v}_{c_2} = v_{t_2}\underline{\hat{e}}_t + v_{n_2}\underline{\hat{e}}_n + v_{b_2}\underline{\hat{e}}_b$$

$$\underline{v}_{c_1}' = v_{t_1}'\underline{\hat{e}}_t + v_{n_1}'\underline{\hat{e}}_n + v_{b_1}'\underline{\hat{e}}_b$$

$$\underline{v}_{c_2}' = v_{t_2}'\underline{\hat{e}}_t + v_{n_2}'\underline{\hat{e}}_n + v_{b_2}'\underline{\hat{e}}_b$$

At the instant of collision, if we draw the FBD of RB ① & ②, we will have a normal reaction force N along the $\underline{\hat{e}}_n$ direction and the impulse will be $\int N\,dt$



Impulse-momentum diagram
(non-impulsive forces that have negligible impulses are not shown)

A finite impulse will cause change in velocity in the direction of impulse

Note that the two RBs colliding are assumed unconstrained

For unconstrained colliding bodies, the equations turn out to be slightly easier.

Using Impulse-momentum relation for RB 1 and 2

$$m_1 \underline{v}_{c_1}' - m_1 \underline{v}_{c_1} = -\int N dt \ \hat{\underline{e}}_n$$

$$m_2 \underline{v}_{c_2}' - m_2 \underline{v}_{c_2} = \int N dt \ \hat{\underline{e}}_n$$

← impulse only along $\hat{\underline{e}}_n$

There is no impulse along $\hat{\underline{e}}_t$ and $\hat{\underline{e}}_b$ directions

⇒ conserve LINEAR MOMENTUM for each body separately in $\hat{\underline{e}}_t$ and $\hat{\underline{e}}_n$ directions:

RB 1	RB 2
$v_{t_1} = v_{t_1}'$ — (1)	$v_{t_2} = v_{t_2}'$ — (3)
$v_{b_1} = v_{b_1}'$ — (2)	$v_{b_2} = v_{b_2}'$ — (4)

[4 equations]

Note: These are velocity components of respective COMs

Now, if we consider the system = 'RB ① + RB ②'

⇒ Impulse $\int N\,dt\ \hat{e}_n$ will be an internal impulse for the entire system ('RB ① + RB ②')

Therefore, for the system subjected to only non-impulsive external forces, the linear momentum is conserved for the system during the internal impulsive interval. Therefore,

$$\boxed{\underbrace{m_1 v_{n_1} + m_2 v_{n_2}}_{\text{before collision}} = \underbrace{m_1 v_{n_1}' + m_2 v_{n_2}'}_{\text{after collision}}} \quad \text{— ⑤}$$

Next, consider the angular impulse of the two RBs about pt 'O' fixed in the inertial frame.

$$\rightarrow \quad \underline{I}_{ang\,O} = \int \cancelto{\underline{0}}{\underline{M}_O^{imp}}\,dt = \underline{0} \quad (\text{for both RBs})$$

Since impulsive force $N\hat{e}_n$ passes through pt 'O', the net resultant impulsive moment $\underline{M}_O^{imp} = \underline{0}$

⇒ Angular momentum is conserved for the two RB separately

'O' is fixed in frame 'I' $\Rightarrow$ 'O' is a valid pt for $\underline{M}_O = \underline{\dot{H}}_{O|I}$

RB ① : $\underline{H}_O = \underline{H}'_O$ (pt 'O' coincides with pt 'A')

$$\Rightarrow \underline{H}_{c_1} + \underline{r}_{c_1O} \times m\,\underline{v}_{c_1} = \underline{H}'_{c_1} + \underline{r}_{c_1O} \times m\,\underline{v}'_{c_1}$$

['|I' frame reference dropped for ease of writing]

$$\Rightarrow \underbrace{\underline{H}_{c_1}}_{\underline{\underline{I}}^{c_1}\,\underline{\omega}_1} = \underbrace{\underline{H}'_{c_1}}_{\underline{\underline{I}}^{c_1}\,\underline{\omega}'_1} + \underline{r}_{c_1O} \times m_1 \underbrace{(\underline{v}'_{c_1} - \underline{v}_{c_1})}_{(v'_{n_1} - v_{n_1})\,\hat{\underline{e}}_n}$$

only V_{n_1} changes after impact

$$\Rightarrow \boxed{\underline{\underline{I}}^{c_1}\,\underline{\omega}_1 = \underline{\underline{I}}^{c_1}\,\underline{\omega}'_1 + \underline{r}_{c_1O} \times m\left[(v'_{n_1} - v_{n_1})\,\hat{\underline{e}}_n\right]}$$

Equate $\hat{\underline{e}}_n$, $\hat{\underline{e}}_t$, and $\hat{\underline{e}}_b$ components on both sides and we get three equations $\rightarrow$ 6, 7, 8

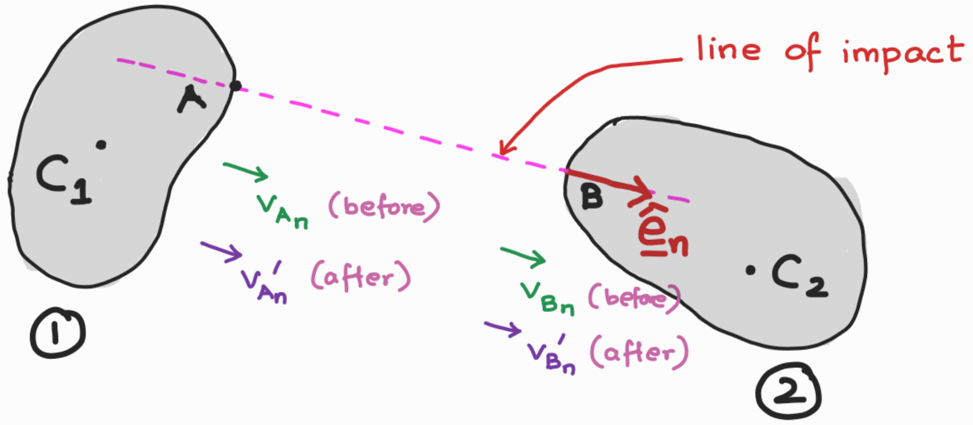
RB ② : Just like RB 1, we can proceed similarly and we get three more equations for RB ② $\rightarrow$ 9, 10, 11

$$\Rightarrow \boxed{\underline{\underline{I}}^{c_2}\,\underline{\omega}_2 = \underline{\underline{I}}^{c}\,\underline{\omega}'_2 + \underline{r}_{c_2O} \times m\left[(v'_{n_2} - v_{n_2})\,\hat{\underline{e}}_n\right]}$$

The 12^{th} equation comes from through the use of the

Coefficient of Restitution (e) !

e → an empirical parameter which models the dissipation of energy during impact



defined as

$$e \equiv \frac{\text{velocity of separation of contact pts}}{\text{velocity of approach of contact pts}}$$

along normal direction ($\hat{e}_n$)

experimentally determined

(Range $0 \leq e \leq 1$)

$$\Rightarrow \boxed{e = -\frac{(v'_{B_n} - v'_{A_n})}{(v_{B_n} - v_{A_n})}} \quad \text{— (12)}$$

Note: The above formula involves contact point velocities and not COM velocities and they can be expressed in terms of the velocities of the COMs as:

$$v_{A_n} = \underline{v}_A \cdot \hat{\underline{e}}_n = (\underline{v}_{C_1} + \underline{\omega}_1 \times \underline{r}_{AC_1}) \cdot \hat{\underline{e}}_n$$

$$v'_{A_n} = \underline{v}'_A \cdot \hat{\underline{e}}_n = (\underline{v}'_{C_1} + \underline{\omega}'_1 \times \underline{r}_{AC_1}) \cdot \hat{\underline{e}}_n$$

[∵ 'A' and 'C_1' are both points on RB 1] and

$$v_{B_n} = \underline{v}_B \cdot \hat{\underline{e}}_n = (\underline{v}_{C_2} + \underline{\omega}_2 \times \underline{r}_{BC_2}) \cdot \hat{\underline{e}}_n$$

$$v'_{B_n} = \underline{v}'_B \cdot \hat{\underline{e}}_n = (\underline{v}'_{C_2} + \underline{\omega}'_2 \times \underline{r}_{BC_2}) \cdot \hat{\underline{e}}_n$$

[∵ 'B' and 'C_2' are both points on RB ②]

What happens when RB ② is massive? ($m_2 \to \infty$)

$m_2 \to \infty \Rightarrow \int N\,dt$ does not affect RB ②

$\Rightarrow \underline{\omega}_2' \approx \underline{\omega}_2$ and $\underline{v}'_{C_2} \approx \underline{v}_{C_2}$

$\Rightarrow$ ONLY $\underbrace{\text{6 unknowns}}$ in this case

$\underline{\omega}_1'$ and $\underline{v}'_{C_1}$ of RB ① after impact are unknown

1) Smooth impact

$$\Rightarrow \quad v_{t_1} = v'_{t_1} \quad \text{and} \quad v_{b_1} = v'_{b_1} \qquad \text{(2 eqns)}$$

2) Linear momentum conservation for entire system

$$m_1 v_{n_1} + m_2 v_{n_2} = m_1 v'_{n_1} + m_2 v'_{n_2}$$

$$\Rightarrow \quad m_1 (v_{n_1} - v'_{n_1}) = \underbrace{\cancel{m_2}^{\infty} \underbrace{(v'_{n_2} - v_{n_2})}_{\approx 0}}_{\text{finite but unknown}} \quad \times \quad \text{Not useful}$$

3) Angular momentum conservation abt coincident point 'O' fixed in 'I' frame separately for two RBs

RB ①: $\underline{H}_{c_1} + \underline{r}_{c_1 O} \times m_1 \underline{V}_{c_1} = \underline{H}_{c_1}' + \underline{r}_{c_1 O} \times m_1 \underline{V}_{c_1}'$

(3 eqns)

RB ②: $\underbrace{\underline{H}_{c_2}}_{\underline{\underline{I}}^{C_2}\,\underline{\omega}_2} + \underline{r}_{c_2 O} \times m_2 \underline{V}_{c_2} = \underbrace{\cancel{\underline{H}_{c_2}'}}_{\underline{\underline{I}}^{C_2}\,\underline{\omega}_2} + \underline{r}_{c_2 O} \times m_2 \cancel{\underline{V}_{c_2}'}$

$\underline{H}_{c_2}$ $\quad$ $\underline{V}_{c_2}$

since $\underline{\omega}_2' \approx \underline{\omega}_2$

$\Rightarrow$ no equations obtained from RB ②

4) Relating contact point velocities via coefficient of restitution

$$e = -\frac{V_{B_n}' - V_{A_n}'}{V_{B_n} - V_{A_n}}$$

We have $V_{B_n}' \approx V_{B_n}$ ($\because$ B is the pt of impact of massive RB ②)

$$\Rightarrow e = -\frac{(V_{B_n} - V_{A_n}')}{(V_{B_n} - V_{A_n})}$$

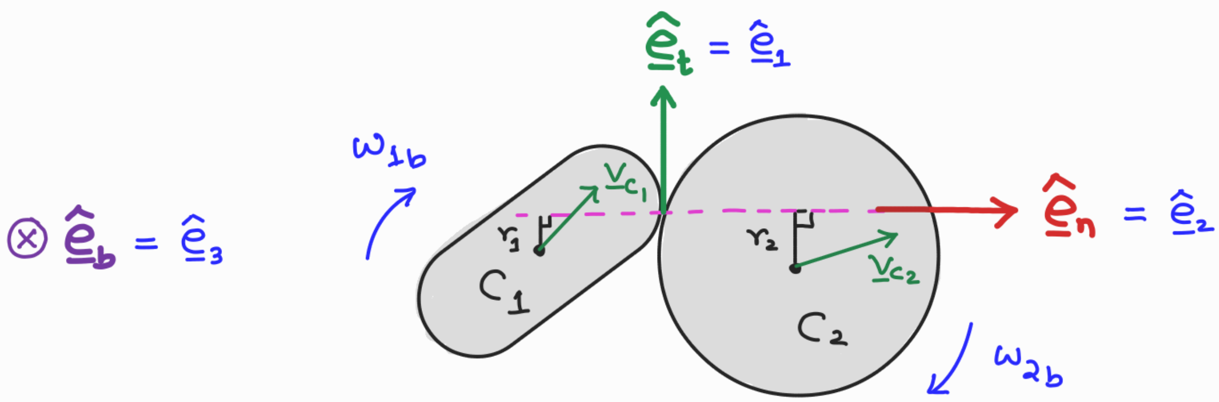
(1 eqn)

So we get total 6 equations for 6 unknowns

Smooth Planar Impact

In this case, the two RBs ① and ② maintain planar motion ($\hat{\underline{e}}_n - \hat{\underline{e}}_t$ plane) before and after impact

→ this is possible if and only if C_1 and C_2 lie in the $\hat{\underline{e}}_n - \hat{\underline{e}}_t$ plane AND $\hat{\underline{e}}_b$ is a principal axis at C_1 for RB ① AND at C_2 for RB ②



Note: $\underline{\omega}_1'$ and $\underline{\omega}_2'$ have components ONLY in $\hat{\underline{e}}_b$ direction

Unknowns: v_{n_1}', v_{t_1}', ω_{1b}' (for RB 1 after impact)

v_{n_2}', v_{t_2}', ω_{2b}' (for RB 2 after impact)

$\Rightarrow$ 6 unknowns (instead of 12 unknowns)

We need to get 6 equations to solve for 6 unknowns:

1> Smooth impact $\Rightarrow$ no impulse along $\hat{\underline{e}}_t$ direction

$$v_{t_1} = v_{t_1}' \quad \text{and} \quad v_{t_2} = v_{t_2}' \longrightarrow (1) \ \& \ (2)$$

2> Conservation of linear momentum for entire system (RB (1) + RB (2)) along $\hat{\underline{e}}_n$

$$m_1 v_{n_1} + m_2 v_{n_2} = m_1 v_{n_1}' + m_2 v_{n_2}' \longrightarrow (3)$$

3> Conservation of angular momentum about coincident point 'O' fixed in inertial frame 'I' for each body

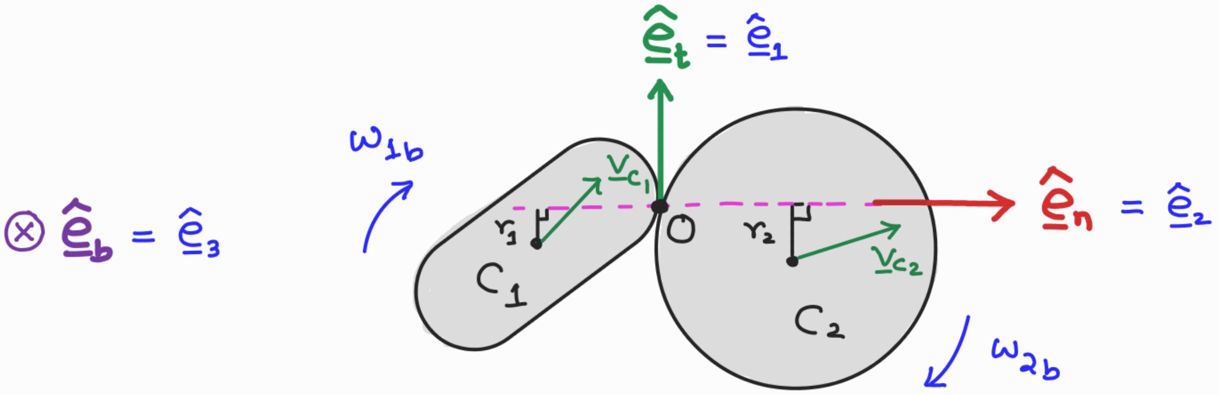
RB (1) : $$\underset{\underset{\underline{\underline{I}}_c\,\underline{\omega}_1}{\parallel\!|}}{\underline{H}_{c_1}} + \underline{r}_{c_1 O} \times m_1 \underline{v}_{c_1} = \underset{\underset{\underline{\underline{I}}_c\,\underline{\omega}_1'}{\parallel\!|}}{\underline{H}_{c_1}'} + \underbrace{\underline{r}_{c_1 O} \times m_1 \underline{v}_{c_1}'}_{r\hat{\underline{e}}_1 \times m\left[v_{t_1}'\hat{\underline{e}}_1 + v_{n_1}'\hat{\underline{e}}_2\right]}$$

(i) $$[\underline{\omega}_1]_{(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)} = \begin{bmatrix} 0 \\ 0 \\ \omega_{b_1} \end{bmatrix} ; \quad [\underline{\omega}_1']_{(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)} = \begin{bmatrix} 0 \\ 0 \\ \omega_{b_1}' \end{bmatrix}$$

(ii) $\hat{\underline{e}}_3 = \hat{\underline{e}}_b$ is a principal axis at C_1

$$\Rightarrow I_{23}^{C_1} = I_{13}^{C_1} = 0$$

simplifies computation of $\underline{\underline{I}}_c\,\underline{\omega}_1$ and $\underline{\underline{I}}_c\,\underline{\omega}_1'$



RB ① : $\underline{H}_{c_1} + \underline{r}_{c_1 O} \times m_1 \underline{v}_{c_1} = \underline{H}'_{c_1} + \underbrace{\underline{r}_{c_1 O} \times m_1 \underline{v}'_{c_1}}_{r\hat{\underline{e}}_1 \times m\left[v'_{t_1}\hat{\underline{e}}_1 + v'_{n_1}\hat{\underline{e}}_2\right]}$

$\underline{H}_{c_1} \equiv \underline{\underline{I}}_c\, \underline{\omega}_1 \qquad \underline{H}'_{c_1} \equiv \underline{\underline{I}}_c\, \underline{\omega}'_1$

$$\Rightarrow \quad I_{33}^{c_1}\,\omega_{b_1} + m_1 v_{n_1} r_1 = I_{33}^{c_1}\,\omega'_{b_1} + m_1 v'_{n_1} r_1 \qquad (4)$$

+ or − sign will depend on the outcome of the cross-product

RB ② : Doing similarly for RB 2, we get another equation:

$$I_{33}^{c_2}\,\omega_{b_2} + m_2 v_{n_2} r_2 = I_{33}^{c_2}\,\omega'_{b_2} + m_2 v'_{n_2} r_2 \qquad (5)$$

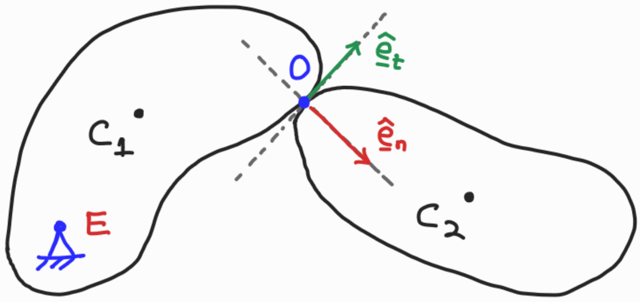
4) Relating contact point velocities through coefficient of restitution:

$$v'_{B_n} - v'_{A_n} = -e\left(v_{B_n} - v_{A_n}\right) \qquad \text{— } (6)$$

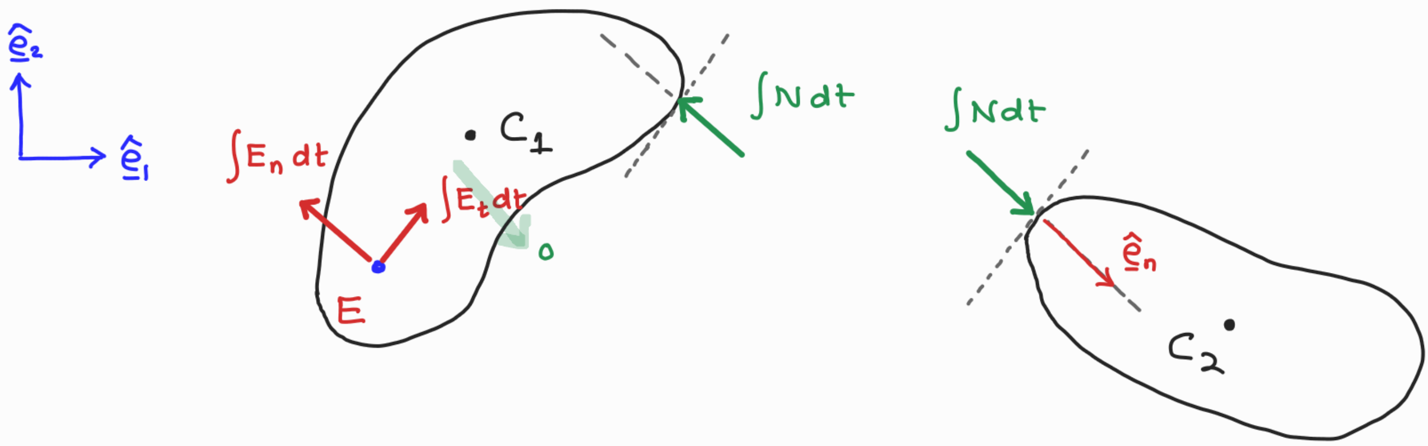
We now have 6 eqns and 6 unknowns!

Collision of RB with constraints

If one or both of the colliding RBs is constrained to rotate about a point E (see figure below), an impulsive reaction is also exerted at E during impact at O

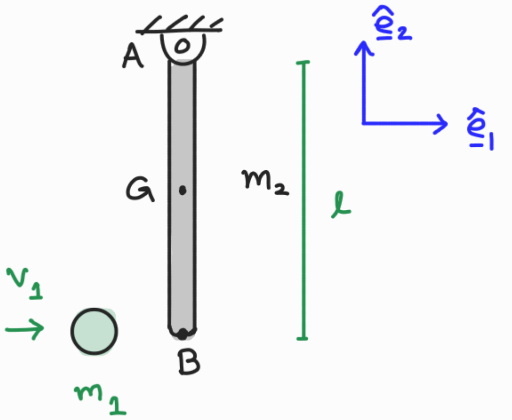


Impulse-momentum diagram



In these cases of constrained collisions, it is better to write out momentum-impulse equations for each RBs separa-tely and treat the impulse forces unknown (and to be found) as well.

Let's do an example to understand this idea!



Planar 2D problem : Ball (treated as a particle) strikes the lower end of rod AB suspended from a hinge at A and is initially at rest. $e = 0.8$. Find angular velocity of rod and velocity of ball immediately after impact.

Solu: Unknowns:

Particle velocity after impact : V_1'

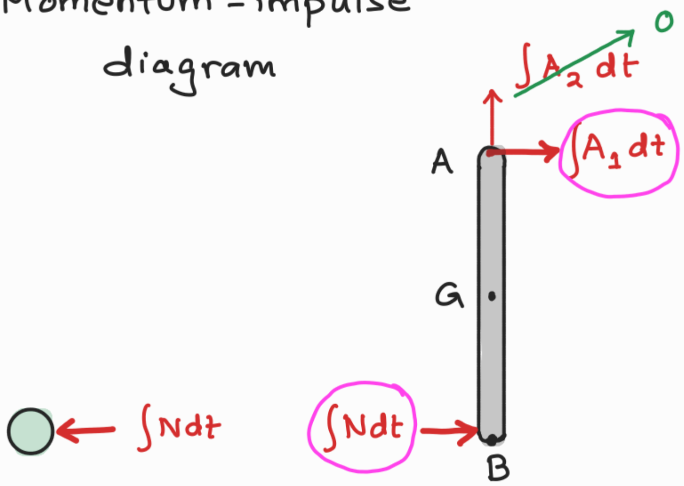
Rod COM velocity after " : $\underline{V}_G'$

Rod angular " " " : $\underline{\omega}_{AB}'$

Since the system is planar $\Rightarrow$ $\underline{\omega}_{AB}' = \omega_3' \hat{\underline{e}}_3$

$$\underline{V}_G' = V_{G_1} \hat{\underline{e}}_1 + V_{G_2} \hat{\underline{e}}_2$$

Momentum - impulse diagram



(Impulse only in $\hat{e}_1$ direction)

5 UNKNOWNS

1) Smooth collision: No velocity change along $\hat{\underline{e}}_2 = \hat{\underline{e}}_t$ direction

Rod: $V_{G_2}' = V_{G_2} = 0$ — (1)

2> Impulse-momentum relations along $\hat{e}_1 = \hat{e}_n$ direction

Particle: $$m_1 v_1' - m_1 v_1 = \int N\,dt \qquad — (2)$$

Rod : $$m_2 v_{G_1}' - m_2 \cancelto{0}{v_{G_1}} = \int N\,dt + \int A_1\,dt \qquad — (3)$$

initially at rest

3> Angular impulse - angular momentum relations along $\hat{e}_3$

(since planar)

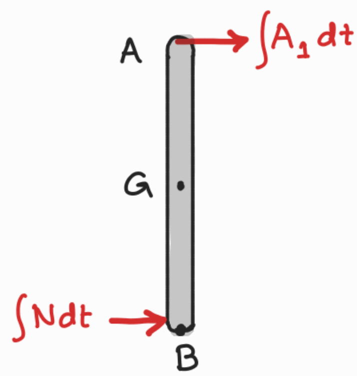
Rod abt pt A : $$H_{A_3}' - H_{A_3} = l_{AB}\left(\int N\,dt\right)$$

$H_{A_3}' \to I_{33}^A\,\omega_3'$ $\qquad$ $H_{A_3} \to I_{33}^A\,\omega_3$

fixed to inertial frame

'I' ⇒ 'A' is a valid pt for

$M_A = \dot{H}_{A|I}$



$$\Rightarrow \int N\,dt = \frac{I_{33}^A\,\omega_3' - I_{33}^A\,\cancelto{0}{\omega_3}}{l_{AB}} \qquad — (4)$$

4> Relation between contact point velocities using coefficient of restitution

$$v_{B_2}' - v_1' = -e\left(\cancelto{0}{v_{B_2}} - v_1\right) \qquad — (5)$$

5 Unknowns can be solved using 5 equations