Recap : In the last lecture, we introduce definitions of reference frame, how it can be attached to a rigid body, coordinate system embedded in the reference frame with origin O, definitions of position vector, velocity vector and acceleration vector useful for defining motion (kinematics) of a point w.r.t a reference frame F.

Position vector, velocity, and accelerations in various csys relative to fixed reference frame

In this lecture, our goal is to derive the velocity and acceleration in three types of coordinate systems

a) Cartesian csys (x_1, x_2, x_3) $\rightarrow$ origin & orientation remains fixed at all times

b) Cylindrical polar csys (r, ϕ, z)

c) Path csys (s)

$\rightarrow$ origin remains fixed but orientations may change with time

Kinematics of a Particle/Point in Cartesian Csys

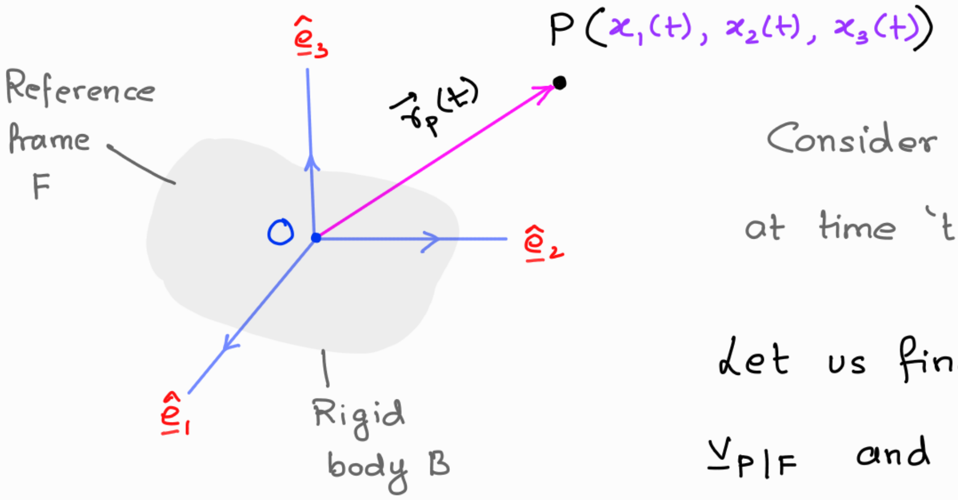
(or frame-embedded)

We consider a body-fixed Cartesian csys with 3 orthogonal unit vectors

means the origin O and $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ are fixed to frame F

The Cartesian csys is embedded in F, hence it does not vary with time in relation to F.

$$\Rightarrow \quad \frac{d}{dt}\hat{e}_1\Big|_F = 0, \quad \frac{d}{dt}\hat{e}_2\Big|_F = 0, \quad \frac{d}{dt}\hat{e}_3\Big|_F = 0$$



Consider the location of moving pt P at time 't': $(x_1(t), x_2(t), x_3(t))$

Let us find the expressions for $\underline{v}_{P|F}$ and $\underline{a}_{P|F}$

The position vector of moving pt. P w.r.t F :

$$\underline{r}_P(t) = x_1(t)\,\hat{e}_1 + x_2(t)\,\hat{e}_2 + x_3(t)\,\hat{e}_3$$

The velocity vector

$$\underline{v}_{P|F} = \frac{d}{dt}\left(\underline{r}_P\right)\Big|_F = \frac{d}{dt}\left(x_1\,\hat{\underline{e}}_1 + x_2\,\hat{\underline{e}}_2 + x_3\,\hat{\underline{e}}_3\right)\Big|_F$$

$$= \frac{dx_1}{dt}\,\hat{\underline{e}}_1 + x_1 \cancelto{0}{\frac{d\hat{\underline{e}}_1}{dt}}\Big|_F + \frac{dx_2}{dt}\,\hat{\underline{e}}_2$$

$$+ x_2 \cancelto{0}{\frac{d\hat{\underline{e}}_2}{dt}}\Big|_F + \frac{dx_3}{dt}\,\hat{\underline{e}}_3 + x_3 \cancelto{0}{\frac{d\hat{\underline{e}}_3}{dt}}\Big|_F$$

because
csys is fixed to F

$$= \frac{dx_1}{dt}\,\hat{\underline{e}}_1 + \frac{dx_2}{dt}\,\hat{\underline{e}}_2 + \frac{dx_3}{dt}\,\hat{\underline{e}}_3$$

$$\therefore \quad \underline{v}_{P|F} = \dot{x}_1\,\hat{\underline{e}}_1 + \dot{x}_2\,\hat{\underline{e}}_2 + \dot{x}_3\,\hat{\underline{e}}_3 \qquad \text{where} \quad \dot{x}_i = \frac{dx_i}{dt}$$

Using similar arguments, we can derive acceleration $\underline{a}_{P|F}$

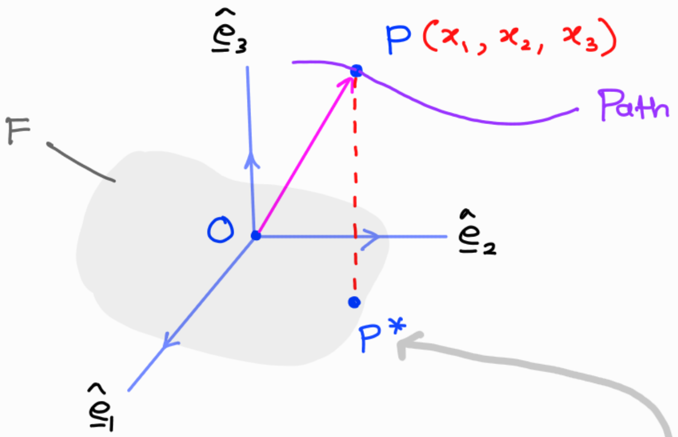
$$\underline{a}_{P|F} = \frac{d}{dt}\left\{\underline{v}_P\big|_F\right\} = \frac{d}{dt}\left\{\dot{x}_1\,\hat{\underline{e}}_1 + \dot{x}_2\,\hat{\underline{e}}_2 + \dot{x}_3\,\hat{\underline{e}}_3\right\}\Big|_F$$

$$= \ddot{x}_1\,\hat{\underline{e}}_1 + \ddot{x}_2\,\hat{\underline{e}}_2 + \ddot{x}_3\,\hat{\underline{e}}_3$$

(where we have once again used

$$\frac{d\hat{\underline{e}}_1}{dt}\Big|_F = \frac{d\hat{\underline{e}}_2}{dt}\Big|_F = \frac{d\hat{\underline{e}}_3}{dt}\Big|_F = 0 \;)$$

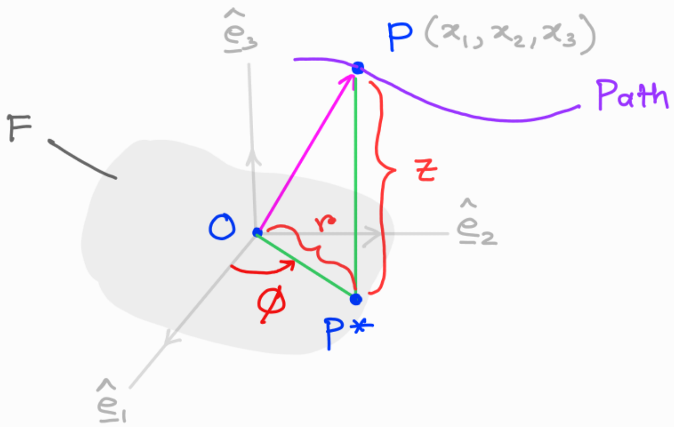
Kinematics of a point in Cylindrical Polar Csys



- Point P is the particle of interest
- Body-fixed Cartesian csys with origin O & three ⊥ unit vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$

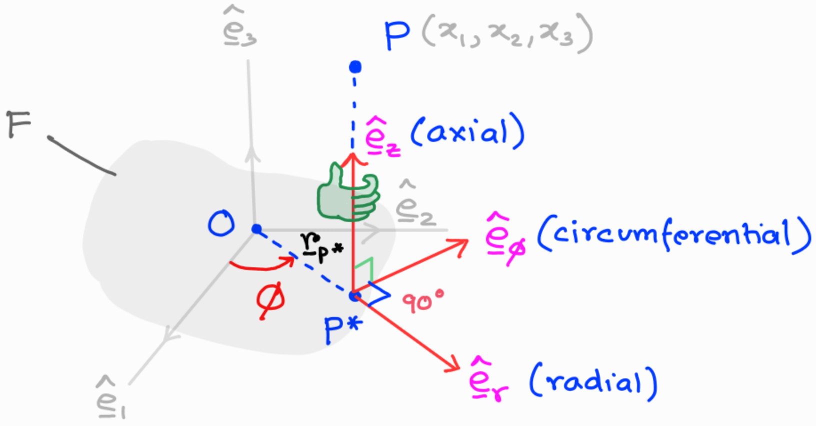
Drop a perpendicular from pt P on the plane spanned by $\hat{e}_1 - \hat{e}_2$

Let's call this projection of P as P^*

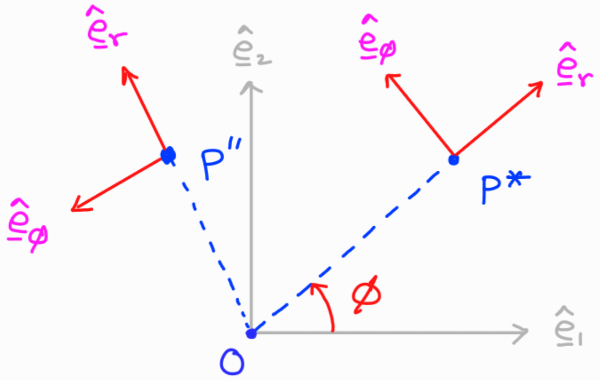


- Connect O to P^* and call the length of $\underline{r}_{P^*}$ as r
- Draw an angle ϕ (+ve as shown going from $\hat{e}_1$ axis to $\hat{e}_2$ axis)
- 'z' is the distance of P from the $\hat{e}_1 - \hat{e}_2$ plane. $z > 0$ along $+\hat{e}_3$

(r, ϕ, z) are the coordinates of a particle P in cylindrical-polar csys. The unit vectors of a cylindrical-polar csys are $\hat{e}_r - \hat{e}_\phi - \hat{e}_z$ (shown in next page)



- $\hat{e}_r$ is along $\underline{r}_{p^*}$
- $\hat{e}_\phi$ is $\perp$ to $\hat{e}_r$
- Both $\hat{e}_r$ and $\hat{e}_\phi$ are parallel to $\hat{e}_1 - \hat{e}_2$ plane
- $\hat{e}_z$ is parallel to $\hat{e}_3$ of the body-fixed Cartesian csys



$\Leftarrow$ The directions of $\hat{e}_r$ and $\hat{e}_\phi$ changes with different locations of P, since ϕ changes, however $\hat{e}_z$ remains same.

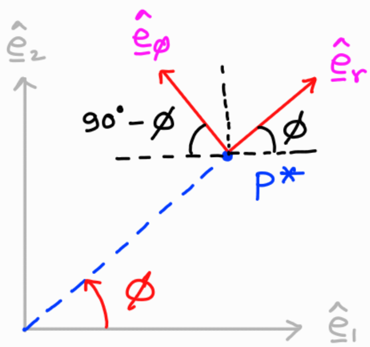
$$\frac{d}{dt} \hat{e}_r \Big|_F \neq 0, \qquad \frac{d}{dt} \hat{e}_\phi \Big|_F \neq 0$$

$$\frac{d}{dt} \hat{e}_z \Big|_F = 0$$

$$\hat{e}_r = \cos\phi \ \hat{e}_1 + \sin\phi \ \hat{e}_2$$

$$\hat{e}_\phi = -\sin\phi \ \hat{e}_1 + \cos\phi \ \hat{e}_2$$

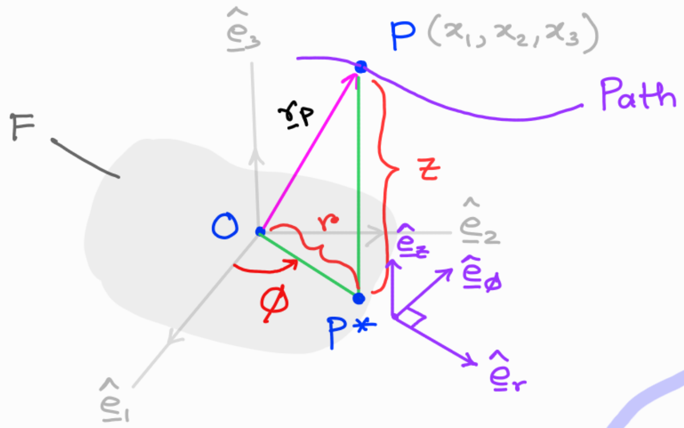
$$\hat{e}_z = \hat{e}_3$$



$$\frac{d}{dt} \hat{e}_r = \frac{d\hat{e}_r}{d\phi} \cdot \underbrace{\frac{d\phi}{dt}}_{\dot{\phi}} = \left(-\sin\phi \ \hat{e}_1 + \cos\phi \ \hat{e}_2\right) \dot{\phi} = \dot{\phi} \ \hat{e}_\phi$$

Similarly, $\quad \dfrac{d}{dt} \hat{e}_\phi = -\dot{\phi} \ \hat{e}_r \quad$ (Verify at home)

Now we want to find the expressions of $\underline{v}_{P|F}$ and $\underline{a}_{P|F}$



$$\vec{OP} = \vec{OP^*} + \vec{P^*P}$$

$$\Rightarrow \underline{r}_P = \underline{r}_{P^*} + z\,\hat{\underline{e}}_z$$

$$= r\,\hat{\underline{e}}_r + z\,\hat{\underline{e}}_z$$

$$= r\left(\cos\phi\,\hat{\underline{e}}_1 + \sin\phi\,\hat{\underline{e}}_2\right) + z\,\hat{\underline{e}}_z$$

① $\underline{r}_P$ depends explicitly on r and z

② $\underline{r}_P$ depends implicitly on ϕ (through $\hat{\underline{e}}_r$)

Velocity of P:

$$\underline{v}_P|_F = \left.\frac{d\underline{r}_P}{dt}\right|_F = \left.\frac{d}{dt}\left(r\,\hat{\underline{e}}_r + z\,\hat{\underline{e}}_z\right)\right|_F$$

$$= \dot{r}\,\hat{\underline{e}}_r + r\,\dot{\hat{\underline{e}}}_r\big|_F + \dot{z}\,\hat{\underline{e}}_z + z\,\cancelto{0}{\dot{\hat{\underline{e}}}_z}\big|_F$$

$$= \dot{r}\,\hat{\underline{e}}_r + r(\dot{\phi}\,\hat{\underline{e}}_\phi) + \dot{z}\,\hat{\underline{e}}_z$$

$$\underline{v}_{P|F} = \underbrace{\dot{r}\,\hat{\underline{e}}_r}_{\text{radial component}} + \underbrace{r\dot{\phi}\,\hat{\underline{e}}_\phi}_{\text{tangential/ circumferential component}} + \underbrace{\dot{z}\,\hat{\underline{e}}_z}_{\text{axial component}}$$

Acceleration :

$$\underline{a}_{P|F} = \frac{d\underline{v}_P}{dt} = \frac{d}{dt}\left(\dot{r}\,\hat{\underline{e}}_r + r\dot{\phi}\,\hat{\underline{e}}_\phi + \dot{z}\,\hat{\underline{e}}_z\right)$$

$$= \ddot{r}\,\hat{\underline{e}}_r + \underbrace{\dot{r}\,\dot{\hat{\underline{e}}}_r}_{\dot{r}\dot{\phi}\,\hat{\underline{e}}_\phi} + \dot{r}\dot{\phi}\,\hat{\underline{e}}_\phi + r\ddot{\phi}\,\hat{\underline{e}}_\phi + \underbrace{r\dot{\phi}\,\dot{\hat{\underline{e}}}_\phi}_{-r\dot{\phi}^2\hat{\underline{e}}_r} + \ddot{z}\,\hat{\underline{e}}_z$$

$$= \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\underline{e}}_r + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\underline{e}}_\phi + \ddot{z}\,\hat{\underline{e}}_z$$

	$\hat{\underline{e}}_r$	$\hat{\underline{e}}_\phi$	$\hat{\underline{e}}_z$
$\underline{r}_{P\|F}$	r	0	z
$\underline{v}_{P\|F}$	$\dot{r}$	$r\dot{\phi}$	$\dot{z}$
$\underline{a}_{P\|F}$	$\ddot{r} - r\dot{\phi}^2$	$r\ddot{\phi} + 2\dot{r}\dot{\phi}$	$\ddot{z}$

Note:

$$\frac{d\underline{r}_{P_\phi}}{d\phi} \neq \underline{v}_{P_\phi}, \text{ etc.}$$

(unlike Cartesian coordinates!)

Kinematics of a Particle/Point in Path coordinates

Thus far, the relations for the motion, velocity, and acc. of a particle have been applied in: (a) Cartesian csys, and (b) Cylindrical polar csys.

However, in some problems, it is more convenient to refer the motion of a particle to a special moving frame that follows the particle along its path through space.

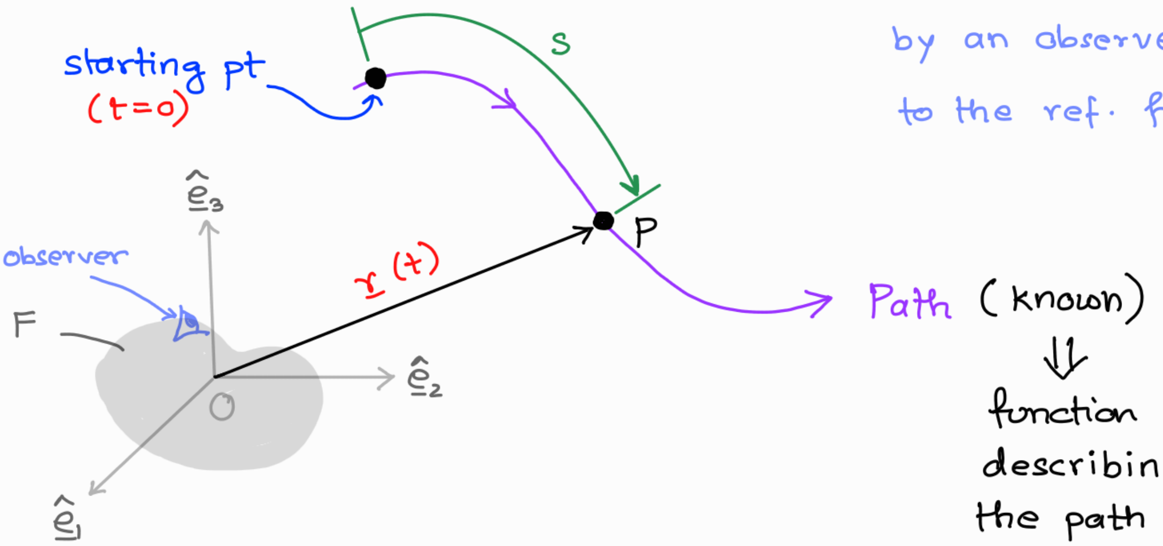
The idea that the motion of a point should be described in terms of properties of its path may not seem obvious. However, this is how we think when using a road map and the speedometer (measures speed) and odometer (measures distance travelled) of a car (say).

This type of description is known as path coordinates, because the basic parameters that are supposed to change are linked to the properties of the path.

↳ The terms tangent and normal components are used because those are the primary directions

We assume that the path of the particle P is known. The most fundamental variable for a specified path is the arc-length 's' along the curve, measured from some starting point to the point of interest

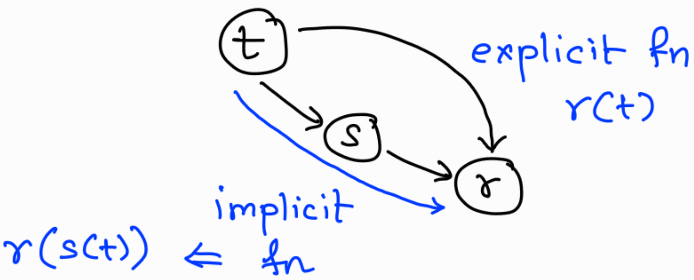
The path 's' is observed by an observer fixed to the ref. frame F



The position vector of point P at time t is given by $\underline{r}(t)$

However, we will define the position vector $\underline{r}$ explicitly as a function of the length 's' of the curve and implicitly on time t

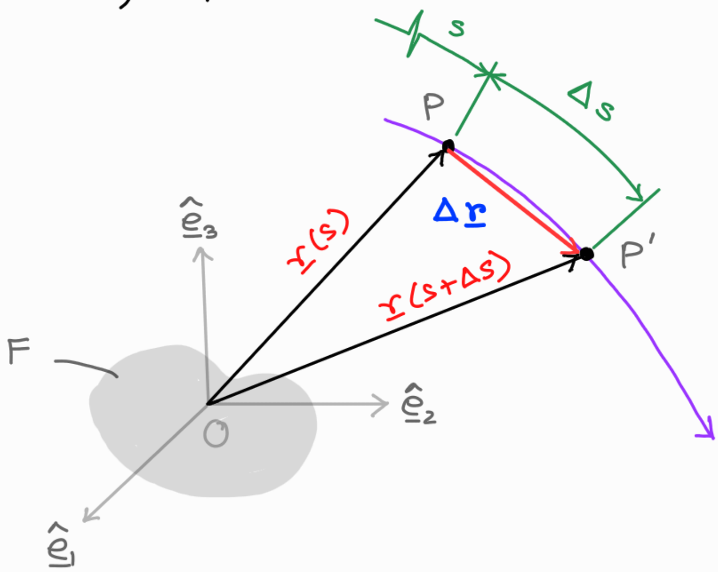
$\Longrightarrow \quad \underline{r}(s(t))$



Next, we derive the formulas for velocity and acceleration which will now involve chain-rule for differentiation

$$\frac{d\underline{r}}{dt} = \frac{d\underline{r}}{ds} \cdot \frac{ds}{dt}$$

But before that lets first derive some basic laws governing geometry of curves



Two position vectors for locations P and P′ separated by a small arc-length Δs

The displacement $\Delta\underline{r}$ is the change in the position of the point P as it moves from position s to s+Δs

$$\Delta\underline{r} = \underline{r}(s+\Delta s) - \underline{r}(s)$$

If Δs is very small, then $|\Delta s| \approx |\Delta\underline{r}|$

(arc) ≈ (st. line)

$$\Rightarrow \lim_{\Delta s \to 0} \left| \frac{\Delta\underline{r}}{\Delta s} \right| = 1 \Rightarrow \left| \frac{d\underline{r}}{ds} \right| = 1$$

$\Rightarrow \dfrac{d\underline{r}}{ds}$ is a unit vector (what is the direction?)

As $\Delta s \to 0$, the direction of $\Delta\underline{r}$ vector approaches tangency to the curve, in the sense of increasing s.

This tangent direction is defined by the unit tangent vector $\underline{\hat{e}}_t$ which is the first unit vector of path coordinates.

$$** \qquad \underline{\hat{e}}_t = \frac{d\underline{r}}{ds}$$

← unit vector as tangent to the path at the point's instantaneous position

This tangent vector is one of the three unit vectors used to describe vector quantities in terms of path coordinates.

The second unit vector in the triad is derived by considering the dependence of $\underline{\hat{e}}_t$ on 's', in the following way:

$$\underline{\hat{e}}_t \cdot \underline{\hat{e}}_t = 1 \quad (\because \text{ unit vector})$$

differentiate w.r.t. 's'

⇓

$$\frac{d\underline{\hat{e}}_t}{ds} \cdot \underline{\hat{e}}_t + \underline{\hat{e}}_t \cdot \frac{d\underline{\hat{e}}_t}{ds} = 0$$

$$\Rightarrow \frac{d\underline{\hat{e}}_t}{ds} \cdot \underline{\hat{e}}_t = 0$$

$$\Rightarrow \frac{d\underline{\hat{e}}_t}{ds} \perp \underline{\hat{e}}_t \quad \Big(\text{if } \underbrace{\frac{d\underline{\hat{e}}_t}{ds} \neq 0}_{\text{trivial case of rectilinear motion}}\Big)$$

The normal direction, with unit vector $\underline{\hat{e}}_n$ is defined parallel to $\frac{d\underline{\hat{e}}_t}{ds}$

Two parallel vectors may be related by a proportionality constant, hence the normal vector $\hat{e}_n$ is defined as:

$$** \qquad \hat{e}_n = \rho \frac{d\hat{e}_t}{ds}$$

dimensions of length

Note $\hat{e}_n$ is a dimensionless unit vector, however $\frac{d\hat{e}_t}{ds}$ has the dimensions of $(\text{length})^{-1}$. Therefore, the proportionality constant must be such that it has the dimensions of length and a magnitude of $\left|\frac{d\hat{e}_t}{ds}\right|^{-1}$

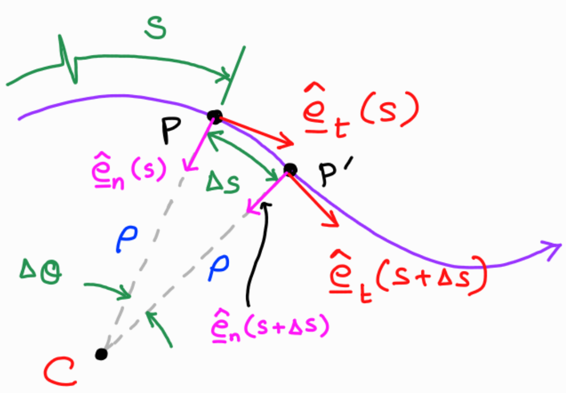
$$\frac{1}{\rho} = \left|\frac{d\hat{e}_t}{ds}\right|$$

has dimensions of length

called the radius of curvature

(why?)

Consider a planar path for understanding "why?". Two points on the path are separated by arclength Δs. We can draw the tangent and normal vectors at those two points.



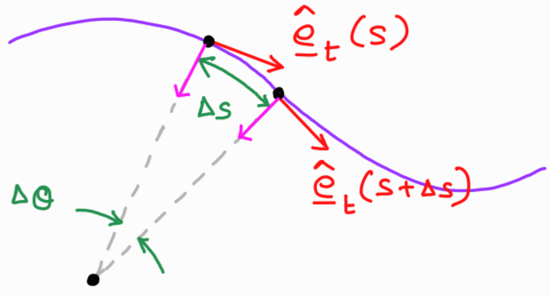
Note the point C is the intersection of the normal vectors at the two close pts. C → local center of curvature

Because Δs is very small, the arc PP' seems like a part of a circle with C as the center and ρ as the radius of the local circle

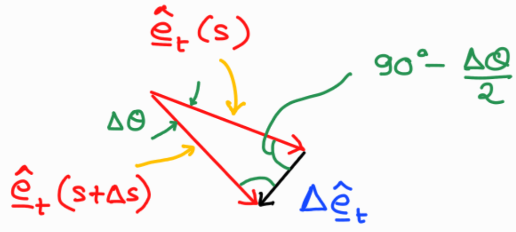
$$\Delta s = \rho\, \Delta\theta \qquad \text{—— (1)}$$

Now consider the increment $\Delta \hat{e}_t$

$$\Delta \hat{e}_t = \hat{e}_t(s+\Delta s) - \hat{e}_t(s)$$



$\Rightarrow$



Isoceles triangle

(since $|\hat{e}_t| = 1$)

As $\Delta s \to 0$, $\Delta\theta \to 0$

As $\Delta\theta \to 0 \Rightarrow \Delta\hat{e}_t \perp \hat{e}_t$

As $\Delta s \to 0$, $|\Delta \hat{e}_t| \to |\cancel{\hat{e}_t}^{\,1}|\, \Delta\theta$

$$\Rightarrow |\Delta \hat{e}_t| \to \Delta\theta \quad — (2)$$

Combining eqns (1) and (2), we get

$$|\Delta\hat{e}_t| \to \frac{\Delta s}{\rho} \quad \Rightarrow \quad \left|\frac{\Delta \hat{e}_t}{\Delta s}\right| \to \frac{1}{\rho}$$

$$\therefore \quad \lim_{\Delta s \to 0} \left|\frac{\Delta \hat{e}_t}{\Delta s}\right| = \left|\frac{d\hat{e}_t}{ds}\right| = \frac{1}{\rho}$$

$$\Rightarrow \quad \underbrace{\left|\rho \frac{d\hat{e}_t}{ds}\right|}_{\hat{e}_n \,\equiv\, \text{normal unit vector}} = 1$$

Hence we showed the idea of ρ being the radius of curvature of the local circle that approximates the path at the instantaneous location of the moving point.

We need one more unit vector to complete the triad.

The third direction is necessary for defining the coordinates of an arbitrary vector in 3D space.

The direction perpendicular to the plane spanned by $\hat{\underline{e}}_t$ and $\hat{\underline{e}}_n$ (also called the osculating plane) is called the **binormal** direction and the corresponding unit vector is $\hat{\underline{e}}_b$.

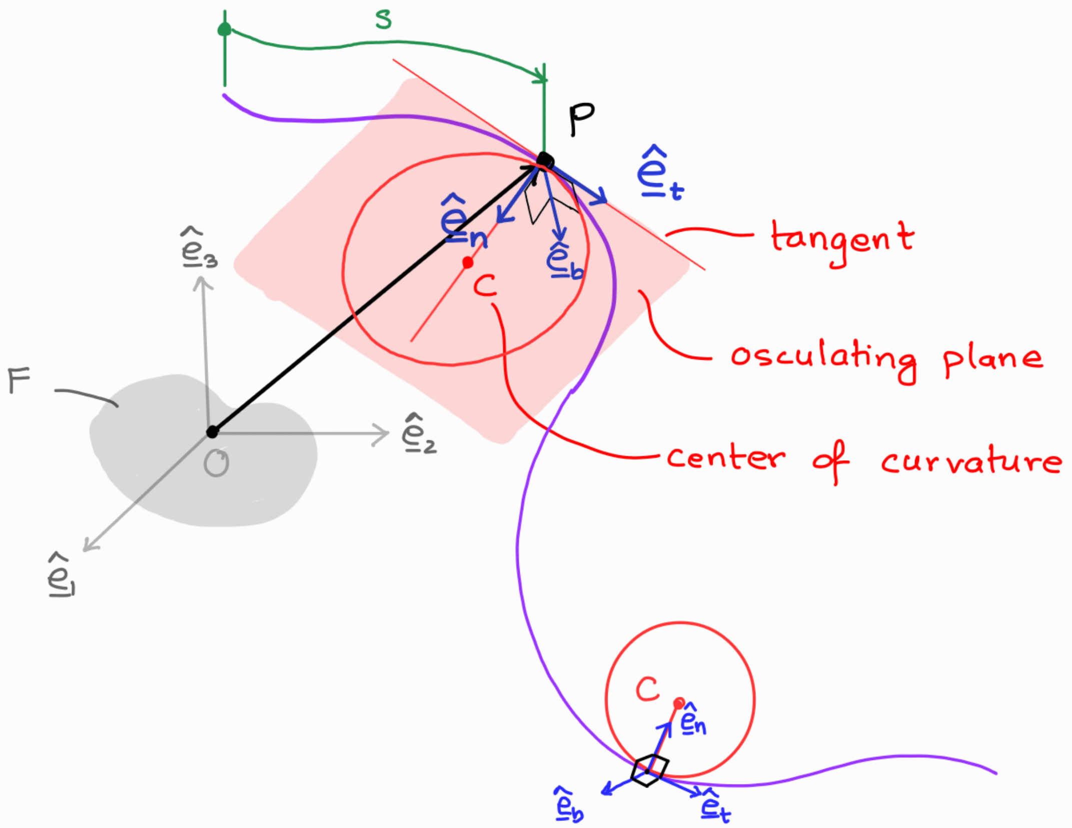
$$** \qquad \hat{\underline{e}}_b = \hat{\underline{e}}_t \times \hat{\underline{e}}_n$$

cross product

normal to both $\hat{\underline{e}}_t$ and $\hat{\underline{e}}_n$

Magnitude of $\hat{\underline{e}}_b$: $\quad |\hat{\underline{e}}_b| = \left| \hat{\underline{e}}_t \times \left(\rho \dfrac{d\hat{\underline{e}}_t}{ds} \right) \right|$

$$= \left| \frac{d\underline{r}}{ds} \times \rho \frac{d^2\underline{r}}{ds^2} \right|$$



$$\hat{\underline{e}}_t = \frac{d\underline{r}}{ds}$$

$$\hat{\underline{e}}_n = \rho \frac{d^2\underline{r}}{ds^2}$$

$$\hat{\underline{e}}_b = \hat{\underline{e}}_t \times \hat{\underline{e}}_n$$

Now we will look to find the expressions of velocity & acceleration of a moving pt using path csys

Velocity in path csys

$$\underline{v}_{P|F} = \left.\frac{d\underline{r}}{dt}\right|_F = \underbrace{\left.\frac{d\underline{r}}{ds}\right|_F}_{\hat{\underline{e}}_t} \cdot \underbrace{\left.\frac{ds}{dt}\right|_F}_{\dot{s}} = \dot{s}\,\underline{e}_t + 0\,\hat{\underline{e}}_n + 0\,\hat{\underline{e}}_b$$

$\dot{s}$ (speed of particle w.r.t. F)

Note $\underline{v}_{P|F}$ has component only along $\hat{\underline{e}}_t$ in path coordinates (as expected)

Acceleration in path csys

$$\underline{a}_{P|F} = \frac{d\underline{v}_{P|F}}{dt} = \frac{d}{dt}(\dot{s}\,\hat{\underline{e}}_t) = \ddot{s}\,\hat{\underline{e}}_t + \dot{s}\,\frac{d\hat{\underline{e}}_t}{dt}$$

$$= \ddot{s}\,\hat{\underline{e}}_t + \dot{s}\,\underbrace{\frac{d\hat{\underline{e}}_t}{ds}}_{\frac{\hat{\underline{e}}_n}{\rho}} \cdot \underbrace{\frac{ds}{dt}}_{\dot{s}}$$

$$= \ddot{s}\,\hat{\underline{e}}_t + \frac{\dot{s}^2}{\rho}\,\hat{\underline{e}}_n + 0\,\hat{\underline{e}}_b$$

$$\underline{a}_{P|F} = \overbrace{\ddot{s}}^{a_t}\,\hat{\underline{e}}_t + \overbrace{\frac{\dot{s}^2}{\rho}}^{a_n}\,\hat{\underline{e}}_n + 0\,\hat{\underline{e}}_b$$

has components along $\hat{\underline{e}}_t$ & $\hat{\underline{e}}_n$

$\ddot{s}\,\hat{\underline{e}}_t$: due to change in magnitude of $\underline{v}_{P|F}$

$\frac{\dot{s}^2}{\rho}\,\hat{\underline{e}}_n$: due to change in direction of $\underline{v}_{P|F}$

Tangential acc. along $\hat{\underline{e}}_t$: $a_t = \ddot{s}$

Centripetal acc. along $\hat{\underline{e}}_n$: $a_n = \frac{\dot{s}^2}{\rho}$

In path csys, acceleration has two components

So far, we derived expressions of $\underline{v}_{P|F}$ and $\underline{a}_{P|F}$ using 3 diff csys

$$\underline{v}_{P|F} = \dot{x}_1 \hat{\underline{e}}_1 + \dot{x}_2 \hat{\underline{e}}_2 + \dot{x}_3 \hat{\underline{e}}_3 \quad \text{(Cartesian csys)}$$

$$= \dot{r} \hat{\underline{e}}_r + r\dot{\phi} \hat{\underline{e}}_\phi + \dot{z} \hat{\underline{e}}_z \quad \text{(Cyl. polar csys)}$$

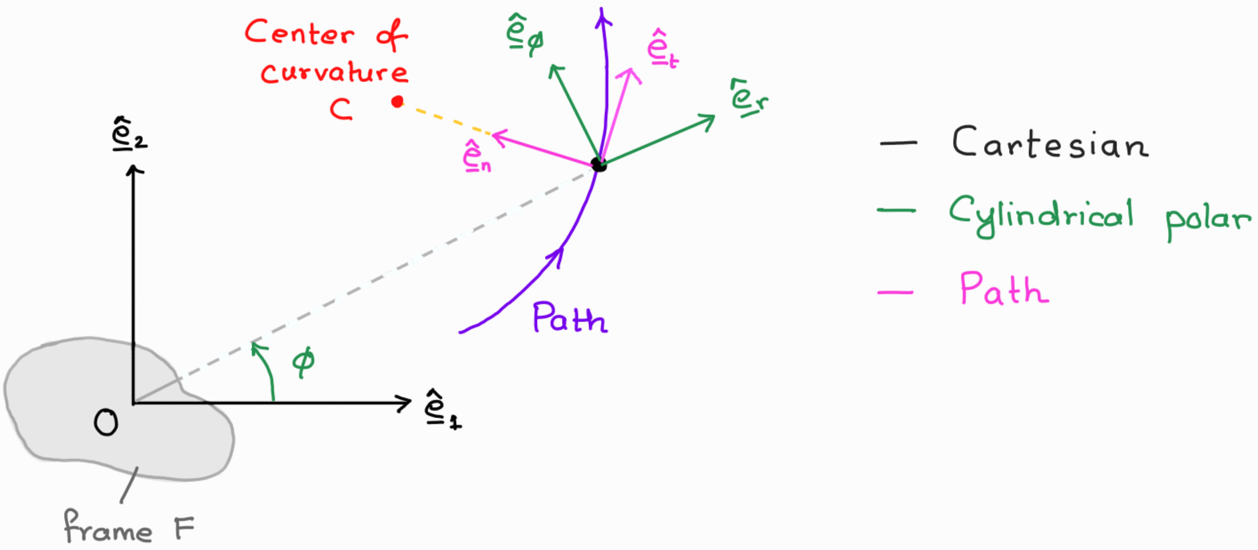
$$= \dot{s} \hat{\underline{e}}_t \quad \text{(path csys)}$$

$$\underline{a}_{P|F} = \ddot{x}_1 \hat{\underline{e}}_1 + \ddot{x}_2 \hat{\underline{e}}_2 + \ddot{x}_3 \hat{\underline{e}}_3 \quad \text{(Cartesian csys)}$$

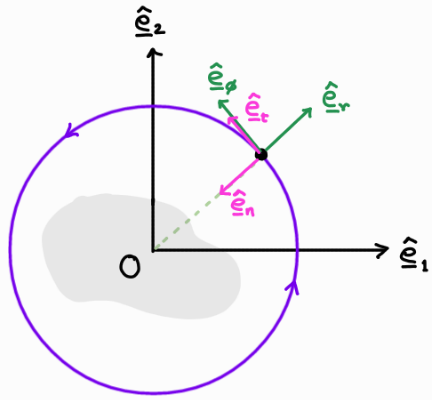
$$= (\ddot{r} - r\dot{\phi}^2) \hat{\underline{e}}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi}) \hat{\underline{e}}_\phi + \ddot{z} \hat{\underline{e}}_z \quad \text{(Cyl. polar csys)}$$

$$= \ddot{s} \hat{\underline{e}}_t + \frac{\dot{s}^2}{\rho} \hat{\underline{e}}_n \quad \text{(path csys)}$$

Suppose a particle P is confined to $\hat{\underline{e}}_1 - \hat{\underline{e}}_2$ plane always, where $O\,\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ is a body-fixed csys. Let us sketch the directions of various unit vectors



Further, if the motion of the particle is along a circle with center at origin O, then



$$\left.\begin{aligned} \hat{\underline{e}}_r &= -\hat{\underline{e}}_n \\ \hat{\underline{e}}_\phi &= \hat{\underline{e}}_t \end{aligned}\right\} \text{ at all times}$$

Also, $\dot{r} = 0$, $\ddot{r} = 0$

Next we derive two useful relationships involving radius of curvature ρ

1) For a general path (or trajectory) in 3D space

$$\boxed{\rho = \frac{|\underline{v}_{P|F}|^3}{|\underline{v}_{P|F} \times \underline{a}_{P|F}|}}$$

2) For a trajectory confined to x-y plane of a body-fixed Cartesian csys $O-\hat{\underline{e}}_1-\hat{\underline{e}}_2-\hat{\underline{e}}_3$ s.t. $\underline{r}_{P|F} = x\,\hat{\underline{e}}_1 + f(x)\,\hat{\underline{e}}_2$, then

$$\boxed{\rho = \frac{\left[1 + \left(\frac{df}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2 f}{dx^2}\right|}}$$

We will only derive the first relation; the second relation is a special case of the first one.

Derivation of $\rho = \dfrac{|\underline{v}_{P|F}|^3}{|\underline{v}_{P|F} \times \underline{a}_{P|F}|}$

Start with $\hat{\underline{e}}_t \times \hat{\underline{e}}_n = \dfrac{d\underline{r}}{ds} \times \rho \dfrac{d^2\underline{r}}{ds^2}$

Take $|\cdot|$ on both sides

$$|\hat{\underline{e}}_t \times \hat{\underline{e}}_n| = |\hat{\underline{e}}_b| = 1 \quad \Rightarrow \quad \left|\frac{d\underline{r}}{ds} \times \rho\frac{d^2\underline{r}}{ds^2}\right| \quad \text{— (A)}$$

Now suppose that the path of particle P is described in a parametric form $\xrightarrow{\text{meaning}}$ if 'α' is some parameter with a range of possible values, then the x_1, x_2, x_3 values (in a body-fixed Cartesian csys $O-\hat{\underline{e}}_1-\hat{\underline{e}}_2-\hat{\underline{e}}_3$) are given in terms of the value of α. In such a situation, the position vector $\underline{r}_{P|F} = \underline{r}$ may be written in component form as:

$$\underline{r} = x_1(\alpha)\,\hat{\underline{e}}_1 + x_2(\alpha)\,\hat{\underline{e}}_2 + x_3(\alpha)\,\hat{\underline{e}}_3$$

With such a description, we can now evaluate the path variables in terms of α

$$\text{(B)} \quad \frac{d\underline{r}}{ds} = \frac{d\underline{r}}{d\alpha} \cdot \frac{d\alpha}{ds}$$

$$\text{(C)} \quad \frac{d^2\underline{r}}{ds^2} = \frac{d}{ds}\left(\frac{d\underline{r}}{d\alpha} \cdot \frac{d\alpha}{ds}\right)$$

$$= \frac{d}{d\alpha}\left(\frac{d\underline{r}}{d\alpha} \cdot \frac{d\alpha}{ds}\right)\frac{d\alpha}{ds}$$

$$= \frac{d^2\underline{r}}{d\alpha^2}\left(\frac{d\alpha}{ds}\right)^2 + \frac{d\underline{r}}{d\alpha}\,\frac{d^2\alpha}{ds^2}$$

Now, putting (B) & (C) in (A):

$$1 = \left|\underbrace{\frac{d\underline{r}}{ds}}_{\text{(B)}} \times \rho\,\underbrace{\frac{d^2\underline{r}}{ds^2}}_{\text{(C)}}\right| = \left|\underbrace{\frac{d\underline{r}}{d\alpha}}_{\text{vector}}\,\underbrace{\frac{d\alpha}{ds}}_{\text{scalar}} \times \rho\left(\underbrace{\frac{d^2\underline{r}}{d\alpha^2}}_{\text{vector}}\underbrace{\left(\frac{d\alpha}{ds}\right)^2}_{\text{scalar}} + \underbrace{\frac{d\underline{r}}{d\alpha}}_{\text{vector}}\,\underbrace{\frac{d^2\alpha}{ds^2}}_{\text{scalar}}\right)\right|$$

$$1 = \left| \rho \left(\frac{d\alpha}{ds}\right)^3 \left(\frac{d\underline{r}}{d\alpha} \times \frac{d^2\underline{r}}{d\alpha^2}\right) + \rho \frac{d\alpha}{ds} \cdot \frac{d^2\alpha}{ds^2} \cancelto{0}{\left(\frac{d\underline{r}}{ds} \times \frac{d\underline{r}}{ds}\right)} \right|$$

$$\frac{1}{\rho} = \left| \left(\frac{d\alpha}{ds}\right)^3 \left(\frac{d\underline{r}}{d\alpha} \times \frac{d^2\underline{r}}{d\alpha^2}\right) \right|$$

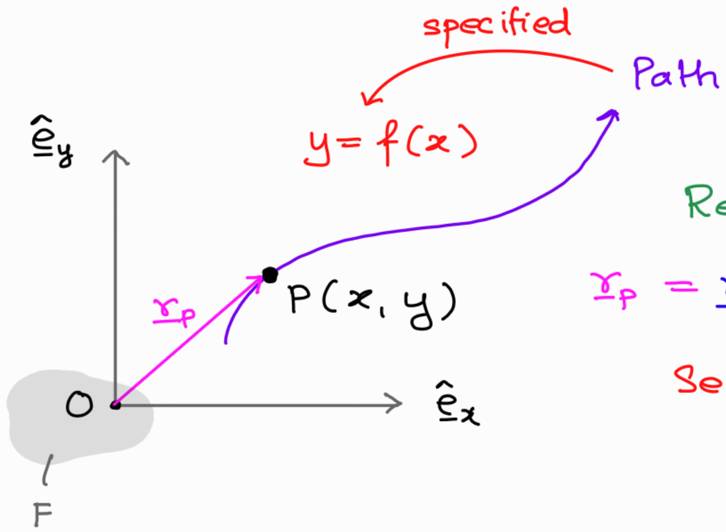
$$\Rightarrow \frac{1}{\rho} = \frac{\left| \left(\frac{d\underline{r}}{d\alpha} \times \frac{d^2\underline{r}}{d\alpha^2}\right) \right|}{\left| \left(\frac{ds}{d\alpha}\right)^3 \right|} \Rightarrow \rho = \frac{\left|\frac{ds}{d\alpha}\right|^3}{\left|\frac{d\underline{r}}{d\alpha} \times \frac{d^2\underline{r}}{d\alpha^2}\right|}$$

Now, say that $\alpha = t$ (time), such that $s = s(t)$, then:

$$\rho = \frac{\left|\frac{ds}{dt}\right|^3}{\left|\frac{d\underline{r}}{dt} \times \frac{d^2\underline{r}}{dt^2}\right|} = \frac{\left|\underline{v}_{P|F}\right|^3}{\left|\underline{v}_{P|F} \times \underline{a}_{P|F}\right|}$$

For 2D planar motion

For a particle having its trajectory confined to $\hat{\underline{e}}_x - \hat{\underline{e}}_y$ plane where $O-\hat{\underline{e}}_x-\hat{\underline{e}}_y$ is a body-fixed Cartesian csys



Recall

$$\underline{r}_P = \underline{r}(\alpha) = x(\alpha)\,\hat{\underline{e}}_x + y(\alpha)\,\hat{\underline{e}}_y$$

Set $\alpha = x$

$$\underline{r} = x\,\hat{\underline{e}}_x + y(x)\,\hat{\underline{e}}_y$$
$$= x\,\hat{\underline{e}}_x + f(x)\,\hat{\underline{e}}_y$$

$$\frac{d\underline{r}}{dx} = \hat{\underline{e}}_x + \frac{df}{dx}\,\hat{\underline{e}}_y \,, \qquad \frac{d^2\underline{r}}{dx^2} = \frac{d^2 f}{dx^2}\,\hat{\underline{e}}_y$$

Using

$$\rho = \frac{\left|\frac{d\underline{r}}{dx}\right|^3}{\left|\frac{d\underline{r}}{dx} \times \frac{d^2\underline{r}}{dx}\right|}$$

$$= \frac{\left(1 + \left[\frac{df}{dx}\right]^2\right)^{3/2}}{\left|\frac{d^2 f}{dx^2}\right|}$$

If 2D planar motion is circular $\Rightarrow$ $\rho = R = \text{constant}$

Radius of circular path