Recap

In the last three lectures, we talked about energy methods, particularly the work-energy principle, that could be used to relate linear and/or angular velocity of an RB at successive instants or locations.

The work-energy thm is a scalar INTEGRAL relation of the Euler's axioms, obtained as integration of Euler's two dynamics equations,

$$W_{1\to 2} = \int_{r_1}^{r_2} P(t)\, dt$$

$$= \int_{r_1}^{r_2} \Big(\underline{F}_R \cdot \underline{v}_{P|I} + \underline{M}_{COM} \cdot \underline{\omega}_{m|I} \Big)\, dt$$

related to Euler's 1st axiom

related to Euler's 2nd axiom

Next, we will consider time integration of the Euler's two vector dynamics equations to get Impulse-Momentum and Angular Impulse- Angular Momentum principles

Let's define impulse and angular impulse

Impulse and Angular Impulse

It is sometimes possible to obtain information about the motion of a body even though a full specification of the resultant force $\underline{F}_R(t)$ and resultant moment $\underline{M}_A(t)$ are not known!

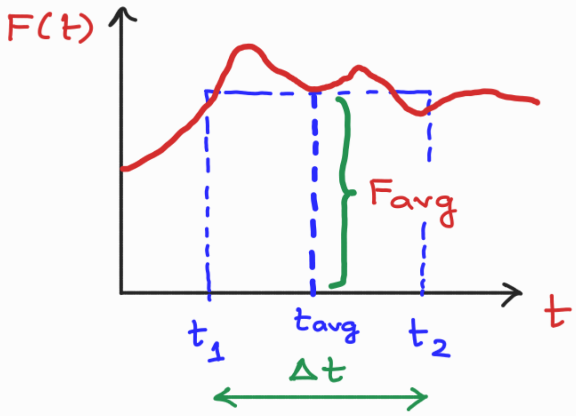
$\underline{F}_R(t)$ written as $\underline{F}(t)$

For example, when a ball strikes a wall, the force exerted by the wall on the ball varies suddenly in time, and though we have no way of knowing the precise way in which this impulsive force changes with time, we can still obtain useful information about the motion of the ball or the force exerted by the wall.

To see how this may be done, lets introduce a vector-valued integral function called the impulse of a force

$$\underbrace{\underline{I}(t_1, t_2)}_{\text{impulse of a resultant force over time interval } t_1 - t_2} = \underbrace{\int_{t_1}^{t_2} \underline{F}(t)\, dt}_{\text{the time integral of the resultant force}} \qquad [\text{Units : N s}]$$

Let's say that $F(t)$ was varying in the fashion shown but is not known to you. Then by mean-value theorem of integral calculus:



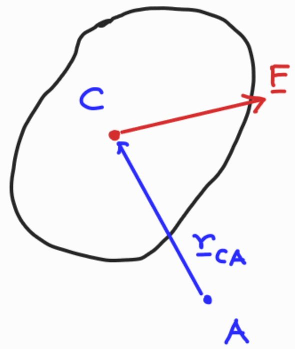
$$\underline{I}(t_1, t_2) = \int_{t_1}^{t_2} F(t)\, dt = \underbrace{\underline{F}_{avg}}_{\text{Average force}} \Delta t \qquad \Delta t = t_2 - t_1$$

$\Rightarrow$ Area under $F(t)$ curve is equal to a constant force F_{avg} acting over time Δt

$$\Rightarrow \quad \underline{F}_{avg} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \underline{F}(t)\, dt = \frac{\underline{I}(t_1, t_2)}{t_2 - t_1}$$

Angular Impulse of force about a pt A

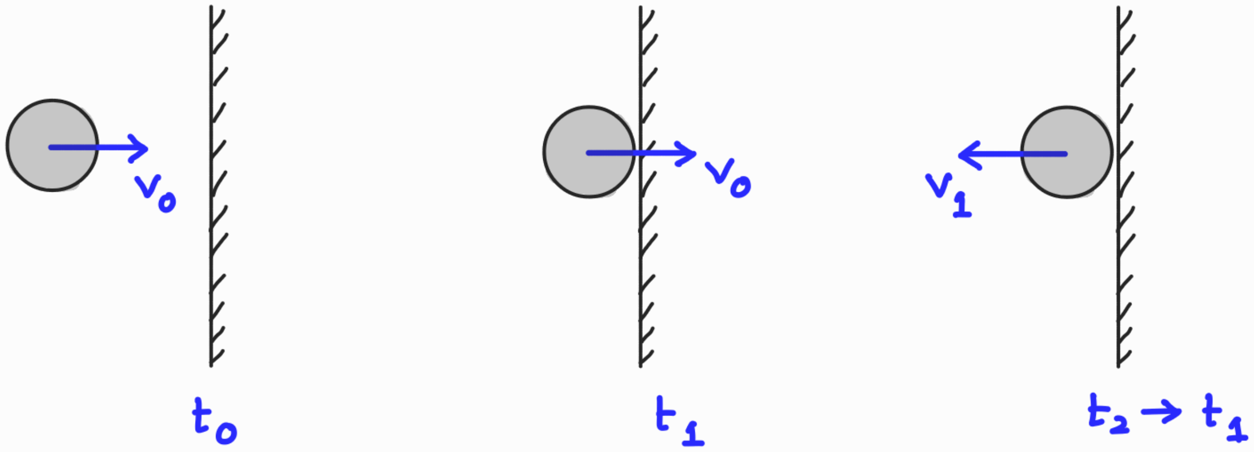
We can defined in a similar fashion the angular impulse of a force abt pt A:



$$\underline{I}_{ang\,A}(t_1, t_2) = \int_{t_1}^{t_2} \underline{M}_A(t)\, dt = \int_{t_1}^{t_2} \underline{r}_{CA} \times \underline{F}(t)\, dt$$

Impulsive force (also called instantaneous impulse)

There are many physical situations in which a change in velocity of a body occurs very suddenly. Take same example of a ball hitting a wall. The ball experiences a finite change in velocity during a very short time



there is no observable change in the position during the impact time $[t_1, t_2]$

In such cases of impact, the impulse occurs instantaneously

This physical idea of an instantaneous impulsive action is characterized by an impulsive force. Mathematically, it is defined as:

Impulsive force :

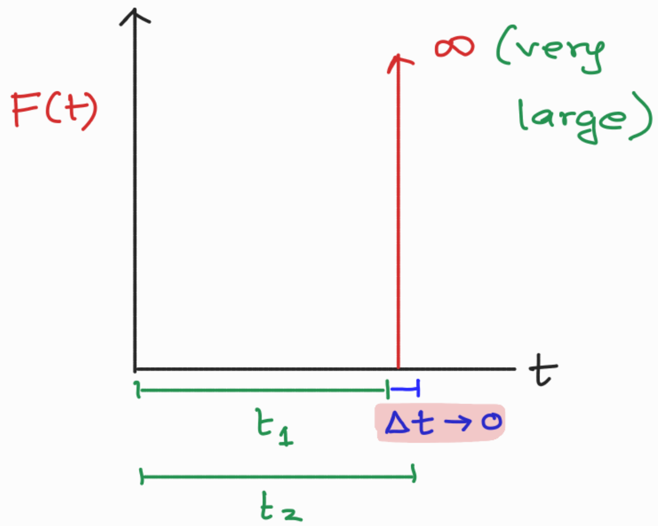
$$\underline{I}(t_1) = \lim_{t_2 \to t_1} \underline{I}(t_1, t_2)$$

(still has dimensions of impulse)

$$= \lim_{t_2 \to t_1} \int_{t_1}^{t_2} \underline{F}^{imp}(t)\, dt = \text{finite}$$

(not zero)

An impulsive force is characterized physically as a very large, suddenly applied force acting over an infinitisimally small time interval, and resulting in an instantaneous but finite change in velocity with no change in position of the body in that duration.



Ideal Impulsive force

$$\int_{t_1}^{t_2} \underline{F}^{imp}(t)\, dt = \text{finite even when } \Delta t \to 0$$

$$\left(\infty \times \text{infinitesimally small } \Delta t = \text{finite} \right)$$

$$\int_{t_1}^{t_2} \underline{F}^{non\text{-}imp}(t)\, dt \approx 0 \quad \text{when } \Delta t \to 0$$

$$\left(\text{finite} \times \text{infinitesimally small } \Delta t \approx 0 \right)$$

Thus, $\displaystyle \lim_{\Delta t \to 0} \int_{t_1}^{t_2} \underline{F}(t)\, dt \approx \int_{t_1}^{t_2} \underline{F}^{imp}(t)\, dt$

Impulsive moment abt pt A

Similarly, one can define impulsive moment abt pt A:

$$\underline{I}_{ang\,A}(t_1) = \lim_{t_2 \to t_1} \underline{I}_{ang\,A}(t_1, t_2) = \lim_{t_2 \to t_1} \int_{t_1}^{t_2} \underline{M}_A^{imp}(t)\, dt$$

and

$$\lim_{\Delta t \to 0} \int_{t_1}^{t_2} \underline{M}_A(t)\, dt \approx \int_{t_1}^{t_2} \underline{M}_A^{imp}(t)\, dt$$

Impulse - Momentum relation

As we said earlier that it is sometimes possible to obtain information about the motion of a body even though a full specification of the resultant force $\underset{\sim}{\underline{F}_R}(t)$ and resultant moment $\underline{M}_A(t)$ are not known!

written as $\underline{F}(t)$

We do this by integrating the Euler's axioms.

Euler's Ist axiom: $$\frac{d}{dt}\{P|_I\}\Big|_I = \underset{\sim}{\underline{F}(t)}$$

Net resultant force on RB

Upon integrating w.r.t time on the interval $[t_1, t_2]$ in an inertial frame, we get impulse - momentum relation:

$$\underline{I}(t_1, t_2) = \int_{t_1}^{t_2} \underline{F}(t)\, dt = \underline{P}(t_2) - \underline{P}(t_1)$$

$$= m\underline{v}_{c|I}(t_2) - m\underline{v}_{c|I}(t_1)$$

Impulse - momentum relation: The impulse of the resultant external force $\underline{F}(t)$ equals the change in momentum. If $\underline{I}(t_1, t_2) = \underline{0}$, then it implies linear momentum is conserved

For instantaneous impulse, i.e., $t_2 \to t_1$ (or $\Delta t \to 0$)

$$\lim_{t_2 \to t_1} \underline{I}(t_1, t_2) = \underline{p}(t_1^+) - \underline{p}(t_1^-)$$

$$\int_{t_1}^{t_2} \underline{F}^{imp}(t)\, dt = m\, \underline{v}_c(t_1^+) - m\, \underline{v}_c(t_1^-)$$

Angular impulse – Angular Momentum relation

Upon integrating Euler's 2nd axiom in an inertial

Euler's 2nd axiom (about arbitrary pt A)

$$\frac{d}{dt}\{\underbrace{\underline{H}_{A|I}(t)}_{\text{angular momentum of RB abt A}}\}\Big|_I = \underbrace{\underline{M}_A(t)}_{\text{Net resultant moment at A}} - m\, \underline{r}_{CA} \times \underline{a}_{A|I}(t)$$

If pt A is a VALID point, then $\frac{d}{dt}\{\underline{H}_{A|I}\}\big|_I = \underline{M}_A$

$\updownarrow$

$\underline{a}_{A|I} = 0$

or

$A \equiv COM$

or

$\underline{a}_{A|I}$ is directed along COM

Integrating the 2nd equation betn time instants $[t_1, t_2]$ for a valid point A:

$$\underline{I}_{ang\,A}(t_1, t_2) = \int_{t_1}^{t_2} \underline{M}_A(t)\, dt = \int_{t_1}^{t_2} \frac{d}{dt}\{\underline{H}_{A|I}\}\Big|_I \, dt$$

$$= \underline{H}_{A|I}(t_2) - \underline{H}_{A|I}(t_1)$$

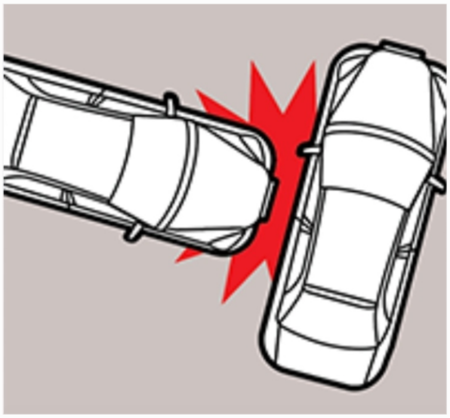
Angular impulse - angular momentum relation: The external angular impulse about pt A equals the change in angular momentum $\underline{H}_{A|I}$. If $\underline{I}_{ang\,A}(t_1, t_2) = \underline{0}$, then angular momentum is conserved, i.e., $\underline{H}_{A|I}(t_1) = \underline{H}_{A|I}(t_2)$

For an instantaneous angular impulse, where $t_2 \to t_1$

$$\lim_{t_2 \to t_1} \underline{I}_{ang\,A}(t_1, t_2) = \underline{H}_{A|I}(t_1^+) - \underline{H}_{A|I}(t_1^-)$$

$$\Rightarrow \int_{t_1}^{t_2} \underline{M}_A^{imp}(t)\, dt = \underline{H}_{A|I}(t_1^+) - \underline{H}_{A|I}(t_1^-)$$

Situations where impulse-momentum relations are useful are in cases of impulsive forces and moments, such as those generated during impacts and explosions



- Impulsive forces/moments induce very large acc to a body over a very short period of time ($\Delta t \to 0$)

- The velocity change (linear and angular) may be evaluated in these cases by considering the time-interval to be so short to only consider the action of impulsive forces/moments

- Further simplifications arise from the fact that only the impulse of the force (avg. value over $[t_1, t_2]$) needs to be known to evaluate the velocities, and not their true time variation.

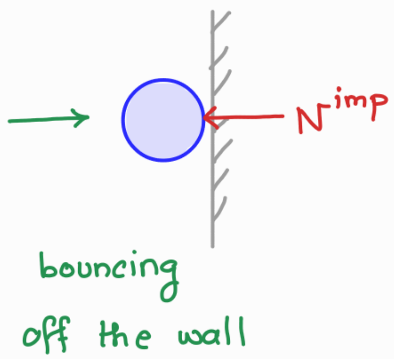
- Impulsive forces/moments are usually large enough that the influence of non-impulsive forces/moment is negligible during the short interval of impulse

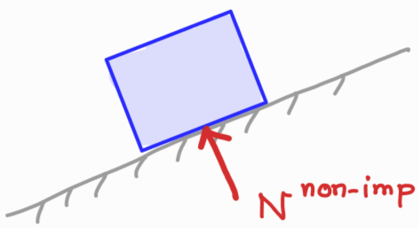
Some other key points:

1> Weight due to gravity is a non-impulsive force — plays no role in impact

2> Spring force is always non-impulsive force — plays no role in impact

3> Normal reaction force may or may not be impulsive — depends upon the constraints on the body



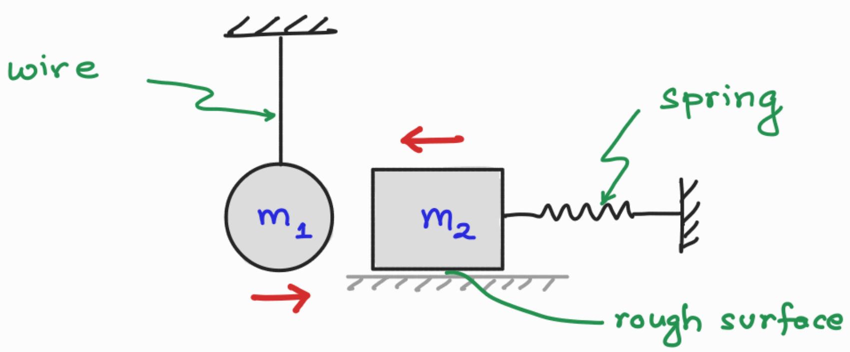


4> Frictional forces, if they exist, would be impulsive if the corresponding normal reaction force is impulsive, otherwise it will be non-impulsive

5) Tension in a rope may or may not be impulsive (depending on the constraints on the body)

6) When in doubt about the nature of a force (tension or normal reaction) being impulsive or not, it is better to draw the FBD of the body in terms of impulses to see if they are actually impulsive or not!

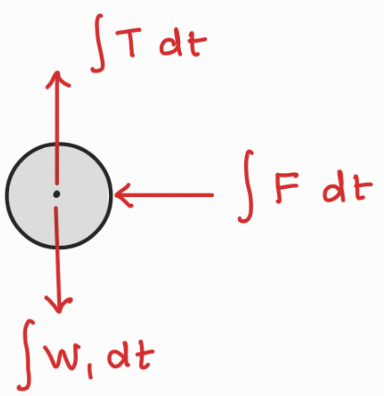
Example : A rigid ball hits a box connected to a spring



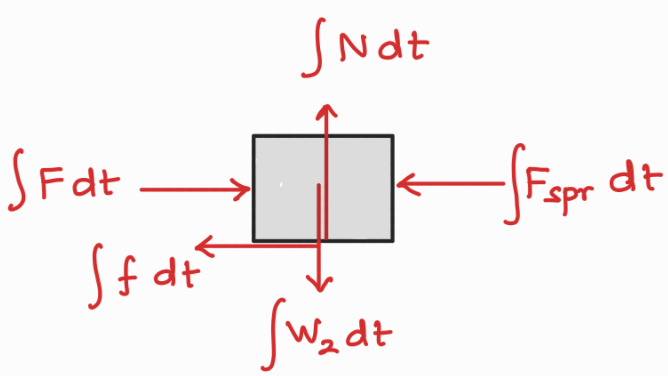
Let's draw FBDs to see which forces are impulsive!

(show in terms of impulses instead of forces; time $[t_1, t_2]$)

FBD (RB 1)



FBD (RB 2)



The impulses that are negligible when $t_2 \to t_1$ ($\Delta t \to 0$)

1) $\int \underbrace{W_1}_{\text{finite}} \, dt = 0$ (as $\Delta t \to 0$)

2) $\int \underbrace{T}_{\text{finite}} \, dt = 0$

3) $\int \underbrace{F}_{\text{very large}} \, dt \neq 0$

4) $\int \underbrace{W_2}_{\text{finite}} \, dt = 0$

5) $\int \underbrace{F_{spr}}_{\text{finite}} \, dt = 0$

6) $\int \underbrace{N}_{\text{finite}} \, dt = 0$

7) $\int \underbrace{f}_{?} \, dt = 0$

8) $\int \underbrace{F}_{\text{very large}} \, dt \neq 0$