Recap

In the previous lecture, we discussed about work-energy principle for RBs subjected to a set of forces. The work-energy principle turned out to be same as was for a particle, i.e.

Equivalent

$$\underbrace{W_{1\to 2}}_{\text{Work done by forces in moving an RB from state 1 to state 2}} = \Delta T = \underbrace{T_2 - T_1}_{\text{Change in the kinetic energy of the RB}} \quad \Big\} \text{ Integral form}$$

or

$$\underbrace{P|_I}_{\text{Mechanical power}} = \underbrace{\frac{d}{dt}\{T|_I\}\Big|_I}_{\text{rate of change of KE}} \quad \Big\} \text{ Differential/ Instantaneous form}$$

Next, we introduced the idea of potential function for a conservative force or the potential energy associated with a conservative force and we stated that a conservative force always admits the following representation:

$$\underbrace{\underline{F}(\underline{r})}_{\text{Conservative force}} = -\frac{d}{d\underline{r}}\underbrace{V(\underline{r})}_{\text{Potential function}}$$

and that the three components of the conservative force $\underline{F}(\underline{r})$ in a csys $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ can be written as:

$$\underline{F}(\underline{r}) = F_1 \hat{\underline{e}}_1 + F_2 \hat{\underline{e}}_2 + F_3 \hat{\underline{e}}_3$$

$$F_1 = -\frac{\partial V}{\partial x_1}, \quad F_2 = -\frac{\partial V}{\partial x_2}, \quad F_3 = -\frac{\partial V}{\partial x_3}$$

Relation between work done by Conservative forces and Potential energy for an RB

If $P_{i|I}$ denotes the power (i.e. rate of work done by a conservative force $\underline{F}_i$ in an inertial frame 'I', then $P_{i|I}$ can be related to the potential function V_i :

$$P_{i|I} = \underline{F}_i \cdot \underline{v}_{i|I}$$

velocity of point 'i' where force $\underline{F}_i$ acts

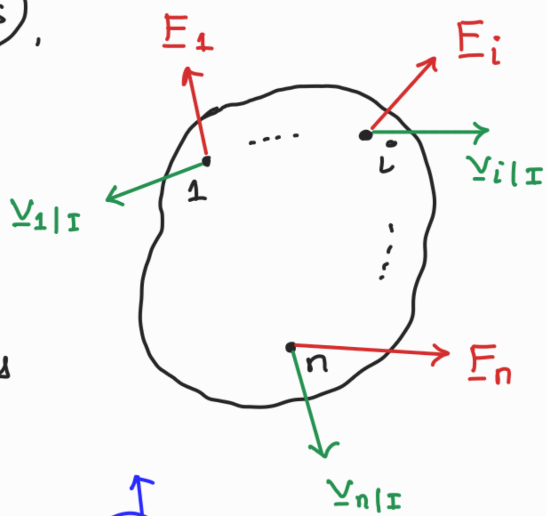
$$\Rightarrow P_{i|I} = \underline{F}_i \cdot \left.\frac{d\underline{r}_i}{dt}\right|_I$$

Recall $\underline{F}(\underline{r}) \cdot d\underline{r} = -dV$

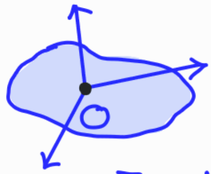
$$\Rightarrow \boxed{P_{i|I} = -\frac{dV_i}{dt}}$$

Potential function associated with conservative force $\underline{F}_i$

If there are 'n' conservative forces acting on an RB (or a system of RBs), then the total power generated would be the sum total of all mechanical power of individual forces



Total mechanical power



Inertial frame 'I'

$$P_{|I} = \sum_{i=1}^{n} P_{i|I}$$

$$= \sum_{i=1}^{n} -\frac{dV_i}{dt}$$

V_i : potential function for the ith conservative force $\underline{F}_i$

$$= -\sum_{i=1}^{n} \frac{dV_i}{dt} \quad \text{— (1)}$$

Integrating (1), we get work done by conservative forces for an RB:

$$\boxed{W_{1\to 2}^{cons} = -\sum_{i=1}^{n} \Delta V_i}$$

So, the total work done by conservative forces acting on an RB is equal to the decrease in the total potential energy

However, we have earlier derived work-energy relation for an RB, where the mechanical power $P|_I$ was related to the time-derivative of the kinetic energy of the RB

Recall $$P|_I = \frac{d}{dt}\{T|_I\}|_I \quad \text{—} ②$$

Equating ① and ②, we get the relation betn kinetic energy and potential energy for conservative forces:

$$-\sum_{i=1}^{n} \frac{dV_i}{dt} = \frac{dT}{dt}$$

Integrate both sides

$$-\sum_{i=1}^{n} \int_{V_{i,1}}^{V_{i,2}} \frac{dV_i}{dt}\, dt = \int_{T_1}^{T_2} \frac{dT}{dt}\, dt$$

$$\Rightarrow -\sum_{i=1}^{n} (V_{i,2} - V_{i,1}) = T_2 - T_1$$

Define $V_2 = \sum_{i=1}^{n} V_{i,2}$ and $V_1 = \sum_{i=1}^{n} V_{i,1}$

$$\Rightarrow -(V_2 - V_1) = T_2 - T$$

$$\Rightarrow \quad -\Delta V = \Delta T$$

$$\Rightarrow \quad \Delta V + \Delta T = 0$$

$$\Rightarrow \quad \Delta(V+T) = 0$$

$$\Rightarrow \quad \boxed{V + T = \text{constant}} \leftarrow \text{Principle of conservation of energy}$$

The total energy (sum of the kinetic and the potential energies) of an RB acted upon by purely conservative forces is constant throughout the motion of the RB.

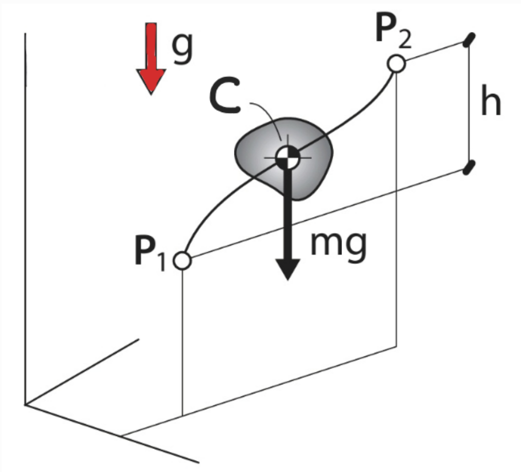
In such cases, if kinetic energy T increases, then potential energy V decreases, and vice versa.

Change in gravitational potential energy of an RB

- A constant force field is always conservative
- Weight of an RB is a conservative force under the assumption of constant gravitational field
- The total gravitational potential energy of an RB in a constant gravitational field is equal to the gravitational potential energy of its COM treated as a particle of equivalent mass of the RB (Proof M.F. Beatty, Eng. Mech. Dynamics Pg 470)

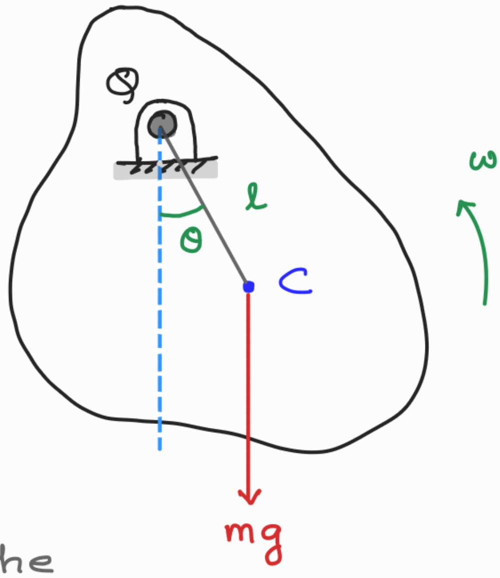
For an RB with mass 'm' and COM C, the potential energy increment between two COM positions with a height difference 'h' is:

$$\Delta V\Big|_1^2 = -W_{1\to2}$$
$$= -(-mgh)$$
$$= mgh$$



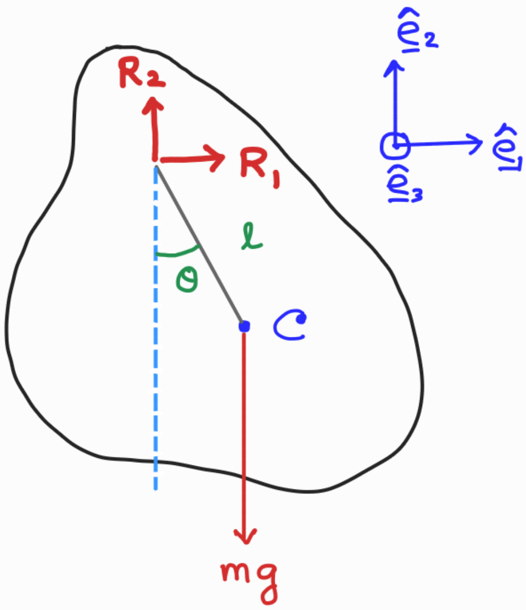
The gravitational PE increases when C goes up (relative to initial position), and decreases when C goes down.

Example: Oscillation of a physical pendulum



A physical pendulum swings about a smooth pin support at Q. The pendulum is released from rest at the placement θ_0 with angular velocity ω_0. Determine the change in the potential energy when $\theta = 0°$.

Soln: Draw FBD first (show all ext. force system)



Reactions arise at the pinned supports → R_1 and R_2. They are workless forces, since point Q does not undergo any motion.

$-mg\,\hat{e}_2$ → is the only force that does non-zero work
→ is a constant gravitational field ⇒ conservative

Change in potential energy from $\theta = \theta_0$ configuration to $\theta = 0°$ configuration:

$$\Delta V \Big|_{\theta=\theta_0}^{\theta=0°} = -mgl\,(1-\cos\theta_0)$$

Change in potential energy of massless linear spring

We consider the linear force

$$\underline{F}_L(\underline{r}) = \alpha \, \underline{r}$$

α constant

Work done by a force $\underline{F}(\underline{r})$ of this form may be written as:

$$dW = \underline{F}_L(\underline{r}) \cdot d\underline{r}$$

and the total work done by $\underline{F}_L(\underline{r})$ in moving a particle over an arbitrary path from a point $\underline{r}_1$ to any other point $\underline{r}_2$ is given by

$$W_{1\to 2} = \int_{\underline{r}_1}^{\underline{r}_2} \underline{F}_L(\underline{r}) \cdot d\underline{r}$$

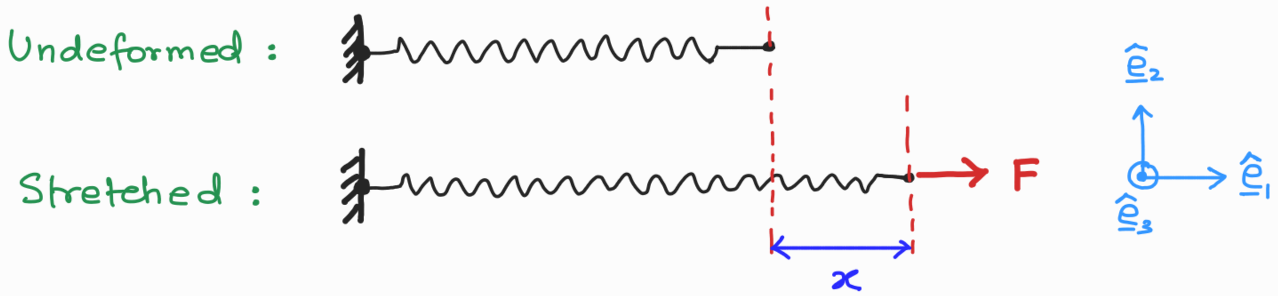
$$= \alpha \int_{\underline{r}_1}^{\underline{r}_2} \underline{r} \cdot d\underline{r}$$

$$= \underbrace{\alpha \left(\underline{r}_2 \cdot \underline{r}_2 - \underline{r}_1 \cdot \underline{r}_1 \right)}_{\text{independent of path and depends only on the initial \& final position}}$$

$$\Downarrow$$

$$\underline{F}_L(\underline{r}) = \alpha \, \underline{r} \Rightarrow \text{Conservative force}$$

Consider a linear spring that undergoes change of length from its undeformed state by a length x



The uniaxial force required to stretch (or compress) the spring is given by $\underline{F} = k x \, \hat{\underline{e}}_1$, which is a linear force of the type $\underline{F}_L(\underline{r}) = \alpha \, \underline{r}$

The work done by the external force in stretching the linear spring from the undeformed state is

$$W_{1\to 2} = \int_0^x \underline{F} \cdot d\underline{r} = \int_0^x k x \, dx = \underbrace{\frac{1}{2} k x^2}_{\text{conservative force}}$$

The internal spring force $\underline{F}_s = -\underline{F} = -k x \, \hat{\underline{e}}_1$ does negative work. Hence, the potential energy stored in a spring in compression or in stretch is $\boxed{\Delta V = \frac{1}{2} k x^2}$

$W_s = -\frac{1}{2} k x^2$

General Energy Principle

In the general case, where a system is acted upon by conservative and non-conservative forces, total mechanical power generated has the form:

$$\underbrace{\text{Total mechanical power } P(t)}_{\substack{= \text{ rate of change} \\ \text{of KE}}} = \underbrace{P^{\text{conservative}}}_{\substack{\text{Mech. power} \\ \text{due to cons.} \\ \text{forces}}} + \underbrace{P^{\text{non-conservative}}}_{\substack{\text{Mech. power due} \\ \text{to non-cons.} \\ \text{forces}}}$$

$$\Rightarrow \quad \frac{dT}{dt} = -\frac{dV}{dt} + P^{\text{non-conservative}}$$

total potential energy

Integrated form of the work-energy relationship

Upon integrating the above differential relation between two time instants t_1 and t_2, we get

$$T_2 - T_1 = -(V_2 - V_1) + W_{1\to 2}^{\text{non-conservative}}$$

$$\Rightarrow \quad \boxed{W_{1\to 2}^{\text{non-conservative}} = (T+V)_2 - (T+V)_1}$$