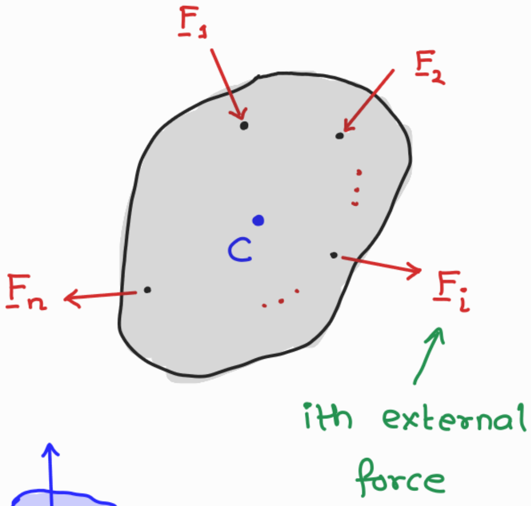
Work - Energy Principle for RBs

Let's see now how the work-energy principle turns out for RBs which are different from particles (as they also have rotation)

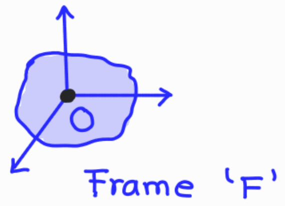
Mechanical Power for an RB

For an RB acted upon by various point ext. forces $\underline{F}_i$ at locations 'i', where the material point velocities are $\underline{v}_{i|I}$



Total mechanical power for the RB

$$= \sum_{i=1}^{n} \underline{F}_i \cdot \underline{v}_{i|F}$$



$$= \sum_{i=1}^{n} \underline{F}_i \cdot \left(\underline{v}_{c|F} + \underline{\omega}_{m|F} \times \underline{r}_{ic} \right)$$

$$= \underbrace{\sum_{i=1}^{n} \underline{F}_i}_{\underline{F}_R \text{ : net resultant of all ext. forces}} \cdot \underline{v}_{c|F} + \sum_{i=1}^{n} \underbrace{\underline{F}_i}_{\underline{a}} \cdot \left(\underbrace{\underline{\omega}_{m|F}}_{\underline{b}} \times \underbrace{\underline{r}_{ic}}_{\underline{c}} \right)$$

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{b} \cdot (\underline{c} \times \underline{a})$$

$$= \underline{F}_R \cdot \underline{v}_{c|F} + \sum_{i=1}^{n} \underbrace{\underline{\omega}_{m|F}}_{\text{const}} \cdot \left(\underline{r}_{ic} \times \underline{F}_i \right)$$

$$= \underline{F}_R \cdot \underline{v}_{c|F} + \underline{\omega}_{m|F} \cdot \underbrace{\left\{ \sum_{i=1}^{n} \left(\underline{r}_{ic} \times \underline{F}_i \right) \right\}}$$

$\underline{M}_c$: Net resultant moment at COM C

$$= \underline{F}_R \cdot \underline{v}_{c|F} + \underline{\omega}_{m|F} \cdot \underline{M}_c$$

Thus, total mechanical power for an RB :

$$P_{|F}(t) = \underbrace{\underline{F}_R(t) \cdot \underline{v}_{c|F}(t)}_{\substack{\text{Power used up} \\ \text{in purely translating} \\ \text{the COM of RB} \\ (\text{assuming } \underline{\omega}_{m|F} = \underline{0})}} + \underbrace{\underline{\omega}_{m|F}(t) \cdot \underline{M}_c(t)}_{\substack{\text{Power used up} \\ \text{in purely rotating the} \\ \text{RB about its COM C} \\ (\text{assuming } \underline{v}_{c|F} = \underline{0})}}$$

Work-Energy Principle for RB

We will relate the mechanical power for an RB to the time rate of change of KE of an RB to derive the work-energy principle for RB.

Show that :

$$P_{|I}(t) = \frac{d}{dt} \left\{ T|_I \right\} \Big|_I$$

We will make use of Euler's two axioms, hence, we will work with the inertial frame of reference 'I'

Recall the KE of an RB computed w.r.t. its COM C

$$T_{|I} = \frac{1}{2} m \underline{v}_{c|I} \cdot \underline{v}_{c|I} + \frac{1}{2} \underline{\omega}_{m|I} \cdot \underline{H}_{c|I}$$

$$= \underbrace{\frac{1}{2} m \underline{v}_{c|I} \cdot \underline{v}_{c|I}}_{A} + \underbrace{\frac{1}{2} \underline{\omega}_{m|I} \cdot \left\{ \int \underline{r}_{pc} \times \underline{v}_{pc|I} \, dm \right\}}_{B}$$

$$B = \frac{1}{2} \underline{\omega}_{m|I} \cdot \int \underline{r}_{pc} \times \underline{v}_{pc|I} \, dm$$

$$= \frac{1}{2} \int \underbrace{\underline{\omega}_{m|I}}_{\underline{a}} \cdot (\underbrace{\underline{r}_{pc}}_{\underline{b}} \times \underbrace{\underline{v}_{pc|I}}_{\underline{c}}) \, dm$$

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{c} \cdot (\underline{a} \times \underline{b})$$

$$= \frac{1}{2} \int \underline{v}_{pc|I} \cdot \underbrace{(\underline{\omega}_{m|I} \times \underline{r}_{pc})}_{\underline{v}_{pc|I}} \, dm$$

$$= \frac{1}{2} \int \underline{v}_{pc|I} \cdot \underline{v}_{pc|I} \, dm$$

$$T_{|I} = A + B$$

$$= \frac{1}{2} m \underline{v}_{c|I} \cdot \underline{v}_{c|I} + \frac{1}{2} \int \underline{v}_{pc|I} \cdot \underline{v}_{pc|I} \, dm$$

Now let's take the time derivative of KE in 'I' frame

$$\frac{d}{dt}\{T|_I\}\Big|_I = m \underbrace{\frac{d}{dt}\{\underline{v}_{c|I}\}\Big|_I}_{\underline{a}_{c|I}} \cdot \underline{v}_{c|I} + \int \underbrace{\frac{d}{dt}\{\underline{v}_{pc|I}\}\Big|_I}_{a_{pc|I}} \cdot \underline{v}_{pc|I}\, dm$$

$$= m\, \underline{a}_{c|I} \cdot \underline{v}_{c|I} + \int \underline{a}_{pc|I} \cdot \underline{v}_{pc|I}\, dm$$

$$= m\, \underline{a}_{c|I} \cdot \underline{v}_{c|I} + \int \underbrace{\underline{a}_{pc|I}}_{\underline{a}} \cdot (\underbrace{\underline{\omega}_{m|I}}_{\underline{b}} \times \underbrace{\underline{r}_{pc}}_{\underline{c}})\, dm$$

$$= m\, \underline{a}_{c|I} \cdot \underline{v}_{c|I} + \int \underline{\omega}_{m|I} \cdot (\underline{r}_{pc} \times \underline{a}_{pc|I})\, dm$$

$$= m\, \underline{a}_{c|I} \cdot \underline{v}_{c|I} + \underline{\omega}_{m|I} \cdot \int \underbrace{\left(\underline{r}_{pc} \times \frac{d}{dt}\{\underline{v}_{pc|I}\}\Big|_I\right)}_{\text{add } \underline{0} = \underline{v}_{pc|I} \times \underline{v}_{pc|I}} dm$$

$$= m\, \underline{a}_{c|I} \cdot \underline{v}_{c|I} + \underline{\omega}_{m|I} \cdot \int \left(\underline{r}_{pc} \times \frac{d}{dt}\{\underline{v}_{pc|I}\}\Big|_I + \frac{d}{dt}\{\underline{r}_{pc}\}\Big|_I \times \underline{v}_{pc|I}\right) dm$$

$$= m\, \underline{a}_{c|I} \cdot \underline{v}_{c|I} + \underline{\omega}_{m|I} \cdot \int \left(\frac{d}{dt}\{\underline{r}_{pc} \times \underline{v}_{pc|I}\}\Big|_I\right) dm$$

$$= m\, \underline{a}_{c|I} \cdot \underline{v}_{c|I} + \underline{\omega}_{m|I} \cdot \frac{d}{dt}\left\{\int \underline{r}_{pc} \times \underline{v}_{pc|I}\, dm\right\}\Big|_I$$

$$= m\, \underline{a}_{c|I} \cdot \underline{v}_{c|I} + \underline{\omega}_{m|I} \cdot \frac{d}{dt}\{\underline{H}_{c|I}\}\Big|_I$$

Since the reference frame is inertial, we can use

Euler's equations:

$$m\,\underline{a}_{C|I} = \underline{F}_R$$

$$\frac{d}{dt}\left\{\underline{H}_{C|I}\right\}\Big|_I = \underline{M}_C$$

is a valid point where $\underline{\dot{H}}_{A|I} = \underline{M}_A$

Using the RHS of these two Euler's equations, we get:

$$\frac{d}{dt}\left\{T_{|I}\right\}\Big|_I = m\,\underline{a}_{C|I}\cdot\underline{v}_{C|I} + \underline{\omega}_{m|I}\cdot\frac{d}{dt}\left\{\underline{H}_{C|I}\right\}\Big|_I$$

$$= \underbrace{\underline{F}_R\cdot\underline{v}_{C|I} + \underline{\omega}_{m|I}\cdot\underline{M}_C}$$

$$= \mathcal{P}_{|I}$$ ≡ Mechanical power used in translating and rotating the RB

Integrating both sides yields the work-energy principle for RBs:

Work-Energy principle: The total work done over a time interval $[t_1, t_2]$ by the external forces and couples acting on an RB about the COM C, in an inertial frame 'I', is equal to the change in the total kinetic energy of the RB:

$$\boxed{W_{1\to 2} = \Delta T = T_2 - T_1}$$

$$\boxed{\frac{d}{dt}\left\{T|_I\right\}\Big|_I = P|_I} \quad \underset{\sim\sim}{\text{OR}} \quad \boxed{W_{1\to 2} = T_2 - T_1}$$

Equivalent work-energy relations

* It is ONE SCALAR equation $\Rightarrow$ can be used to solve for ONLY ONE UNKNOWN

* This one equation is not independent of the six scalar Euler's combined equations

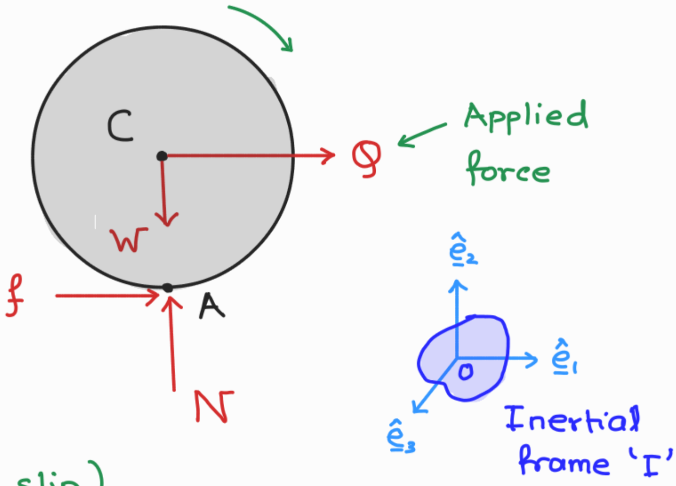
Advantage of Work-Energy Thm over Euler's axioms $\rightarrow$ less algebra, less calculations

(a) Only when ONE unknown needs to be found and if it is possible to find it using work-energy relation, then use this, instead of solving 6 Euler's eqns.

(b) Also, there maybe some external forces acting on the RB, such as the reaction force systems, for which the work done is zero although the forces themselves are non-zero. Such force systems which do not contribute to the work done are called workless force systems. Workless force systems do not appear in the work-energy relation

Example 1 : Disk rolling without slipping

Always draw the FBD of the system being analyzed!



Workless forces :

1> Reaction force N

($\because$ $\underline{v}_{A|I} = \underline{0}$, rolling without slip)

$$(N\hat{e}_2)\cdot(\underline{v}_{A|I}) = 0$$

(Also, N acts in a $\perp$ dir to $\underline{v}_{A|I}$)

2> Frictional force f

($\because$ $\underline{v}_{A|I} = \underline{0}$, rolling without slip)

$$(f\hat{e}_1)\cdot(\underline{v}_{A|I}) = 0$$

3> Weight due to gravity W

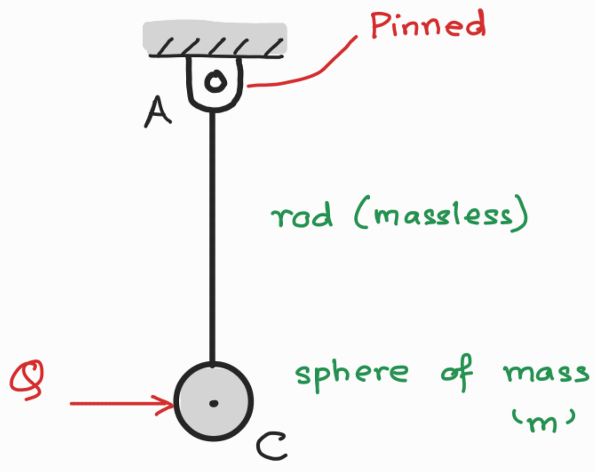
($\because$ W acts in a direction perpendicular to $\underline{v}_{C|I}$)

$$(-W\hat{e}_2)\cdot(v_c\,\hat{e}_1) = 0$$

The only force that does work in this example is force Q

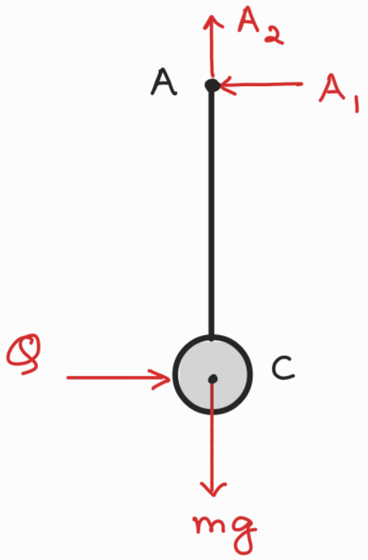
$$(Q\hat{e}_1)\cdot(v_c\,\hat{e}_1) \neq 0$$

Example 2 : Pendulum



Treat system = rod + pendulum

Draw FBD of the system



Workless forces

1) Reaction forces A_1 and A_2

($\because$ $\underline{v}_{A|I} = \underline{0}$ at ALL time instants)

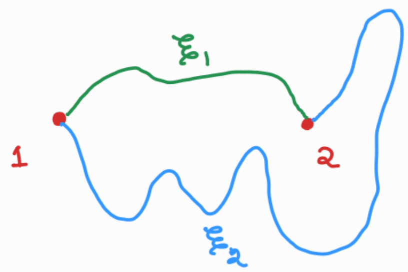
2) Weight due to gravity, mg

($\because$ Weight is $\perp$ to $\underline{v}_{C|I}$ at the instant shown)

Conservative Force & Potential Energy

Recall the defn of conservative forces from previous lecture 16.

Conservative force: A force $\underline{F}(\underline{r})$ acting at a material point is conservative if the work done is independent of the path followed. The net work done only depends on final and initial positions.



$$W_{1-2} = \int_{\underline{r}_1}^{\underline{r}_2} \underline{F}(\underline{r}) \cdot d\underline{r} = \underbrace{S(\underline{r}_2) - S(\underline{r}_1)}_{\text{only a function of initial \& final positions}}$$

Necessary condition for conservative force

Hence if a force $\underline{F}$ is conservative, then it cannot depend explicitly on time or any other variable except the position vector $\underline{r}$ (or spatial coordinates x_1, x_2, x_3)

Non-conservative

$\underline{F}(\underline{r}(t), t)$ ✗

$\underline{F}(t, \underline{a})$ ✗

any other variable

Satisfies necessary cond^n

$\underline{F}(\underline{r}(t))$ ✓

$\underline{F}(x_1(t))$ ✓

Since a conservative force field, $\underline{F}(\underline{r}(t))$, explicitly depends on the position vector $\underline{r}$, conventionally $\underline{F}(\underline{r}(t))$ is defined as conservative if and only if there exists a potential function $V(\underline{r})$ (or potential energy associated with the conservative force) such that

$$W_{1-2} = \int_{\underline{r}_1}^{\underline{r}_2} \underline{F}(\underline{r}) \cdot d\underline{r} = -\int_{\underline{r}_1}^{\underline{r}_2} dV(\underline{r})$$

the negative sign indicates that as a particle goes from a higher potential to a lower potential, the force does positive work on the particle

$$\boxed{dW = -dV}$$

It follows that for a conservative force field, the mechanical work done in moving an RB is equal to decrease in the total potential energy

$$\boxed{\Delta W_{cons} = -\Delta V}$$

Comparing the integrands (the terms under integral):

$$\int \underline{F}(\underline{r}) \cdot d\underline{r} = -\int dV(\underline{r})$$

$$\Rightarrow \boxed{\underline{F}(\underline{r}) = -\frac{dV(\underline{r})}{d\underline{r}}}$$

gradient of a scalar w.r.t a vector yields a vector

$$\Rightarrow \underline{F}(\underline{r}) = -\underbrace{\nabla_{\underline{r}}}_{\text{gradient}} V(\underline{r})$$

$$\nabla_{\underline{r}}(\cdot) \equiv \frac{d}{d\underline{r}}(\cdot)$$

Expressed in a coordinate system $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$:

$$[\underline{r}] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad [\underline{F}(\underline{r})] = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}, \quad \nabla_{\underline{r}} = \begin{bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{bmatrix}$$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = -\begin{bmatrix} \partial V/\partial x_1 \\ \partial V/\partial x_2 \\ \partial V/\partial x_3 \end{bmatrix}$$

or

$$\underline{F} = F_1 \hat{\underline{e}}_1 + F_2 \hat{\underline{e}}_2 + F_3 \hat{\underline{e}}_3 = -\left(\frac{\partial V}{\partial x_1} \hat{\underline{e}}_1 + \frac{\partial V}{\partial x_2} \hat{\underline{e}}_2 + \frac{\partial V}{\partial x_3} \hat{\underline{e}}_3 \right)$$

Furthermore, to detect if a force field $\underline{F}(\underline{r})$ is conservative, it is enough to check simply if the curl of the force field is zero!

$$\underline{\nabla}_{\underline{r}} \times \underline{F}(\underline{r}) = \underline{0} \Rightarrow \text{Force field } \underline{F}(\underline{r}) \text{ is conservative}$$

[Proof of this is not a part of this course]

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \hat{\underline{e}}_1 & \hat{\underline{e}}_2 & \hat{\underline{e}}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Example: If $\underline{F}(r(t)) = F_c$ is a constant force, then it is conservative. You can also easily show that since the force field is constant, the derivatives

$$\frac{\partial F_i}{\partial x_j} = 0 \quad , \quad i,j = 1,2,3.$$ Thus $\underline{\nabla} \times \underline{F} = \underline{0}$!

A common example of a constant conservative force is the weight due to gravity of a body.