Work- Energy principle

So far, we have solved problems by using a combination of Euler's axioms and kinematics. Now, we will explore another approach, in which expressions of work and energy will be derived and used!

- Work-energy methods are alternative methods to understand dynamics of particles / rigid bodies
- We will see that we have to only deal with scalar equation
- Can be effectively used provided force system acting on the particle / RB are conservative (to be defined!)
- This alternative way does not always work so simply

There are different types of energy: mechanical, chemical, nuclear, electrical, magnetic, etc.

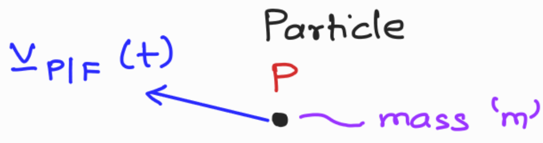
Energy grouping:
- → Kinetic energy (which is due to movement)
- → Potential energy (which are stored energy because of its position or because of its arrangement)

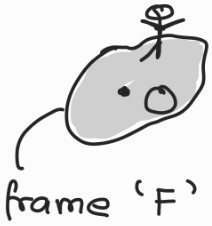
Kinetic Energy

Let's first define KE of a particle and then extend it to RB

Kinetic Energy of a Particle

Kinetic energy of a particle P of mass 'm' moving with velocity $\underline{v}_{P|F}$ relative to a ref. frame 'F':





$$T_{|F} = \frac{1}{2} m \, \underline{v}_{P|F} \cdot \underline{v}_{P|F}$$

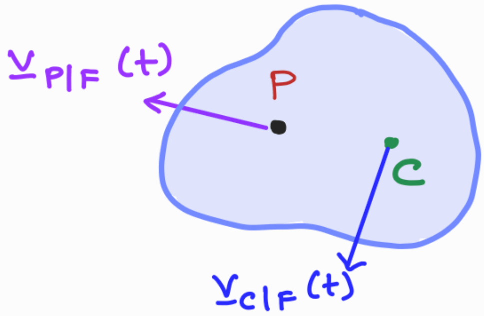
denoted by T (w.r.t. frame 'F')

Kinetic Energy of an RB

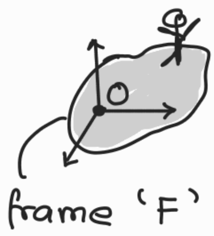
The total KE of a body relative to a (an observer fixed in) ref. frame F is:

$$T_{|F} = \frac{1}{2} \int \underline{v}_{P|F} \cdot \underline{v}_{P|F} \, dm \quad \text{— } (1)$$

velocity of material point P of the RB



$$m = \int dm \equiv \text{total mass of RB}$$

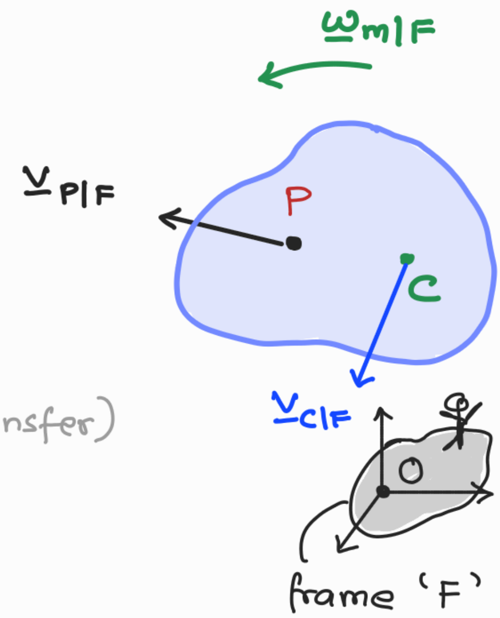


We will next relate the total kinetic energy of the RB to the translational KE of its COM C and the KE of the RB relative to the COM.

Kinetic Energy of an RB computed w.r.t. COM C

The total KE of an RB is equal to the KE of its COM C plus the KE of the RB relative to the COM C.

Let's start from the definition (1) and use the velocity transfer relations for points P and C



$$\underline{v}_{P|F} = \underline{v}_{C|F} + \underline{\omega}_{m|F} \times \underline{r}_{PC} \quad \text{(vel. transfer)}$$

$$\Rightarrow \underline{v}_{PC|F} = \underline{v}_{P|F} - \underline{v}_{C|F} = \underline{\omega}_{m|F} \times \underline{r}_{PC}$$

$$\underline{v}_{P|F} \cdot \underline{v}_{P|F} = \left(\underline{v}_{C|F} + \underline{\omega}_{m|F} \times \underline{r}_{PC}\right) \cdot \left(\underline{v}_{C|F} + \underline{\omega}_{m|F} \times \underline{r}_{PC}\right)$$

$$= \underline{v}_{C|F} \cdot \underline{v}_{C|F} + 2\left(\underline{\omega}_{m|F} \times \underline{r}_{PC}\right) \cdot \underline{v}_{C|F} + \left(\underline{\omega}_{m|F} \times \underline{r}_{PC}\right) \cdot \underbrace{\left(\underline{\omega}_{m|F} \times \underline{r}_{PC}\right)}_{\underline{v}_{PC|F}}$$

$$T_{|F} = \frac{1}{2}\int \underline{v}_{P|F} \cdot \underline{v}_{P|F}\, dm$$

$$= \underbrace{\frac{1}{2}\int \underline{v}_{C|F} \cdot \underline{v}_{C|F}\, dm}_{\text{I}} + \frac{2}{2}\underbrace{\int (\underline{\omega}_{m|F} \times \underline{r}_{PC}) \cdot \underline{v}_{C|F}\, dm}_{\text{II}}$$

$$+ \frac{1}{2}\int \underbrace{(\underline{\omega}_{m|F} \times \underline{r}_{PC}) \cdot \underline{v}_{PC|F}\, dm}_{\text{III}}$$

I $\quad \displaystyle\int \frac{1}{2}\, \underbrace{\underline{v}_{C|F}}_{\text{const. w.r.t intg.}} \cdot \underline{v}_{C|F}\, dm = \frac{m}{2}\, \underline{v}_{C|F} \cdot \underline{v}_{C|F}$

II $\quad \displaystyle\int (\underline{\omega}_{m|F} \times \underline{r}_{PC}) \cdot \underline{v}_{C|F}\, dm = \left\{\underline{\omega}_{m|F} \times \left(\int \underline{r}_{PC}\, dm\right)\right\} \cdot \underline{v}_{C|F}$

const. w.r.t intg.

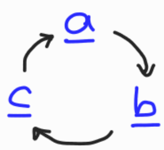
$$\int \underline{r}_{PC}\, dm = \int (\underline{r}_{PO} - \underbrace{\underline{r}_{CO}}_{\text{const. w.r.t intg.}})\, dm = \int \underline{r}_{PO}\, dm - \underline{r}_{CO}\int dm$$

$$= m\underline{r}_{CO} - m\underline{r}_{CO} = 0$$

III $\quad \displaystyle\frac{1}{2}\int (\underbrace{\underline{\omega}_{m|F}}_{\underline{a}} \times \underbrace{\underline{r}_{PC}}_{\underline{b}}) \cdot \underline{v}_{PC|F}\, dm$

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = \underline{a} \cdot (\underline{b} \times \underline{c})$$

[Property



Using this property

$$\frac{1}{2}\int \underbrace{\underline{\omega}_{m|F}}_{\text{const. w.r.t intg}} \cdot (\underline{r}_{PC} \times \underline{v}_{PC|F})\, dm$$

$$= \frac{1}{2}\,\underline{\omega}_{m|F} \cdot \underbrace{\int (\underline{r}_{PC} \times \underline{v}_{PC|F})\, dm}_{\underline{H}_{C|F}}$$

$\underline{H}_{C|F}$ (angular momentum of RB abt C)

$$= \frac{1}{2}\,\underline{\omega}_{m|F} \cdot \underline{H}_{C|F} = \frac{1}{2}\,\underline{\omega}_{m|F} \cdot \left(\underline{\underline{I}}^{C}\,\underline{\omega}_{m|F}\right)$$

Putting together I, II, & III, we get:

$$\boxed{T_{|F} = \frac{1}{2} m\, \underline{v}_{C|F} \cdot \underline{v}_{C|F} + \frac{1}{2}\,\underline{\omega}_{m|F} \cdot \left(\underline{\underline{I}}^{C}\,\underline{\omega}_{m|F}\right)} \quad \text{A}$$

The above expression is the total kinetic energy of the RB in terms of data at COM, and is valid for all reference frames 'F'

Upon using a csys $\hat{\underline{e}}_1$-$\hat{\underline{e}}_2$-$\hat{\underline{e}}_3$, and writing KE in matrix-vector form, we get:

$$T_{|F} = \underbrace{\frac{1}{2} m \overbrace{[\underline{v}_{C|F}]^T [\underline{v}_{C|F}]}^{\text{always} \geq 0}}_{\text{Translational KE of RB}} + \underbrace{\frac{1}{2} \overbrace{[\underline{\omega}_{m|F}]^T [\underline{\underline{I}}^{C}] [\underline{\omega}_{m|F}]}^{\text{always} \geq 0}}_{\text{Rotational KE of RB}}$$

In a principal reference frame where the csys $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$ coincides with the p-axes of the RB

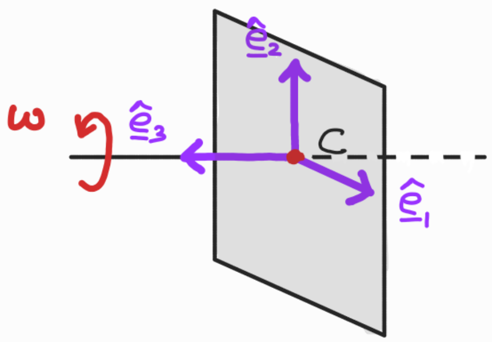
$$[\underline{\underline{I}}^C]_{\begin{pmatrix}\hat{e}_1\\\hat{e}_2\\\hat{e}_3\end{pmatrix}} = \underbrace{\begin{bmatrix} I_{11}^C & 0 & 0 \\ 0 & I_{22}^C & 0 \\ 0 & 0 & I_{33}^C \end{bmatrix}}_{\text{diagonal}}, \qquad [\underline{\omega}_{m|F}]_{\begin{pmatrix}\hat{e}_1\\\hat{e}_2\\\hat{e}_3\end{pmatrix}} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$[\underline{v}_{C|F}]_{\begin{pmatrix}\hat{e}_1\\\hat{e}_2\\\hat{e}_3\end{pmatrix}} = \begin{bmatrix} v_{c,1} \\ v_{c,2} \\ v_{c,3} \end{bmatrix}$$

$$T_{|F} = \frac{1}{2} m \left(v_{c,1}^2 + v_{c,2}^2 + v_{c,3}^2 \right) + \frac{1}{2} \left(I_{11}^C \omega_1^2 + I_{22}^C \omega_2^2 + I_{33}^C \omega_3^2 \right)$$

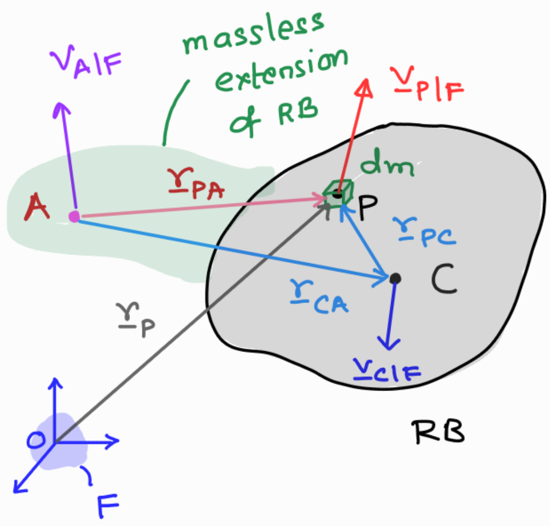
Even further, if the body rotates about a fixed p-axis $\hat{e}_3$, say with $\omega_3 = \omega$ and $I_{33}^C = I$, then the rotational KE reduces even further

$$T_{C|F} = \frac{1}{2} m \left(v_{c,1}^2 + v_{c,2}^2 + v_{c,3}^2 \right) + \frac{1}{2} I_{33}^C \omega^2$$



What if we write KE of RB about a point ≠ COM C ?

KE of RB computed w.r.t. an arbitrary point A of the RB



It is also useful to have a rule for the total KE in terms of motion relative to an arbitrary point A of RB moving with velocity $\underline{v}_{A|F}$

"A" lies on the RB or on its massless extension

KE of the RB relative to pt A

$$T_{|F} = \frac{1}{2}\int \underline{v}_{P|F}\cdot\underline{v}_{P|F}\, dm$$

$$\underline{v}_{P|F} = \underbrace{\underline{v}_{P|F} - \underline{v}_{A|F}} + \underline{v}_{A|F} = \underline{v}_{PA|F} + \underline{v}_{A|F}$$

$$T_{|F} = \frac{1}{2}\int (\underline{v}_{PA|F} + \underline{v}_{A|F})\cdot(\underline{v}_{PA|F} + \underline{v}_{A|F})\, dm$$

$$= \frac{1}{2}\int \underbrace{\underline{v}_{A|F}\cdot\underline{v}_{A|F}}_{\text{const. integrand}} dm + \frac{2}{2}\int \underline{v}_{A|F}\cdot\underline{v}_{PA|F}\, dm$$

$$+ \frac{1}{2}\int \underline{v}_{PA|F}\cdot\underline{v}_{PA|F}\, dm$$

Note: $\underline{v}_{A|F}$ does not depend on the mass distribution of RB

$$= \frac{1}{2} m\, \underline{v}_{A|F}\cdot\underline{v}_{A|F} + \underline{v}_{A|F}\cdot\underbrace{\int \underline{v}_{PA|F}\, dm}_{\text{simplify}} + \underbrace{\frac{1}{2}\int \underline{v}_{PA|F}\cdot\underline{v}_{PA|F}\, dm}_{\text{simplify}}$$

$$\int \underline{v}_{PA|F}\, dm = \int (\underline{v}_{PC|F} + \underbrace{\underline{v}_{CA|F}}_{\text{const.}})\, dm = \int \underline{v}_{PC|F}\, dm + \underline{v}_{CA|F}\int dm$$

$$= \int \underline{v}_{P|F}\, dm - \underline{v}_{C|F}\int dm + \underline{v}_{CA|F}\int dm$$

$$= \cancel{m\,\underline{v}_{C|F}} - \cancel{m\,\underline{v}_{C|F}} + m\,\underline{v}_{CA|F}$$

$$\underline{v}_{PA|F} = \underline{v}_{P|F} - \underline{v}_{A|F} = \underline{\omega}_{m|F} \times \underline{r}_{PA}$$

$$\int \underline{v}_{PA|F} \cdot \underline{v}_{PA|F} \, dm = \int (\underline{\omega}_{m|F} \times \underline{r}_{PA}) \cdot \underline{v}_{PA|F} \, dm$$

Once again use $(\underline{a} \times \underline{b}) \cdot \underline{c} = \underline{a} \cdot (\underline{b} \times \underline{c})$

$$= \int \underbrace{\underline{\omega}_{m|F}}_{\text{const.}} \cdot (\underline{r}_{PA} \times \underline{v}_{PA|F}) \, dm$$

$$= \underline{\omega}_{m|F} \cdot \underbrace{\int \underline{r}_{PA} \times \underline{v}_{PA|F} \, dm}_{\underline{H}_{A|F} \text{ (Angular momentum of RB abt pt A)}}$$

$$= \underline{\omega}_{m|F} \cdot \underline{H}_{A|F}$$

Hence, the total KE of an RB relative to an arbitrary moving pt A in reference frame F is :

$$\boxed{T_{|F} = \frac{1}{2} m \, \underline{v}_{A|F} \cdot \underline{v}_{A|F} + \underline{v}_{A|F} \cdot m \, \underline{v}_{CA|F} + \frac{1}{2} \underline{\omega}_{m|F} \cdot \underline{H}_{A|F}}$$

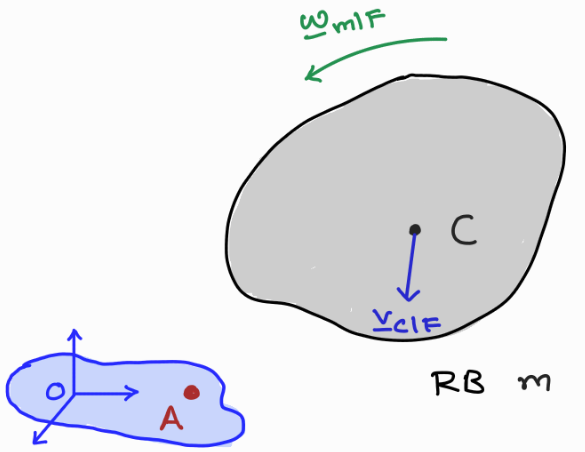
B

Consequently, when point A is fixed in ref. frame F,

$$\underline{v}_{A|F} = \underline{0}$$

and, the total KE of the RB is :

$$\boxed{T_{|F} = \frac{1}{2} \underline{\omega}_{m|F} \cdot \underline{H}_{A|F}}$$



Upon using a csys $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$, and writing KE in matrix-vector form, we get:

$$T_{|F} = \frac{1}{2} \left[\underline{\omega}_{m|F}\right]^T \left[\underline{\underline{I}}^A\right] \left[\underline{\omega}_{m|F}\right]$$

- If a p-axis lies through point A, then the inertia matrix A will be simplified.

- Further, for 2D plane problems, where $\underline{\omega}_{m|F} = \omega \, \hat{\underline{e}}_3$, the KE expression will get simplified.

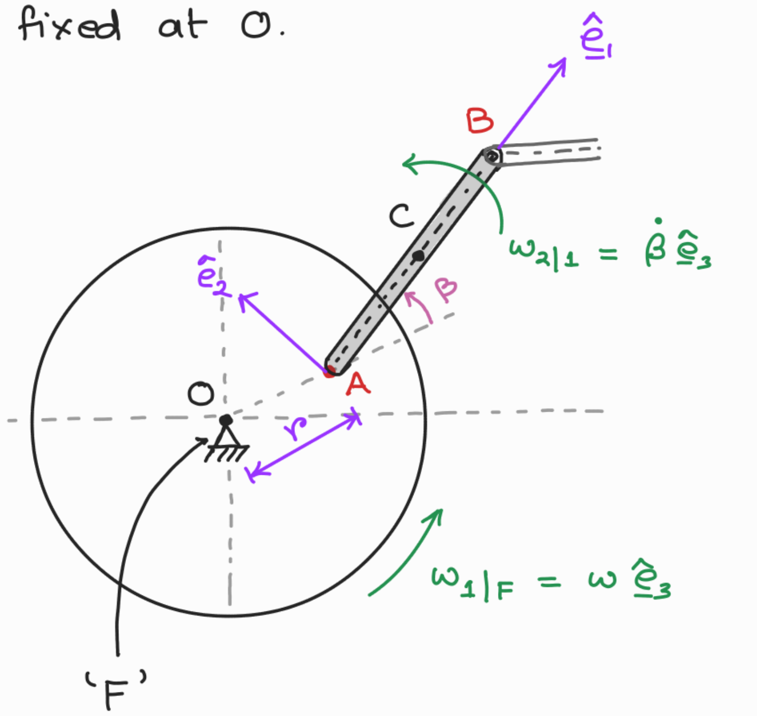
You can further verify that when pt A coincides with the COM C, the expression B reduces to expression A

$$T_{|F} = \frac{1}{2} m \overset{\underline{v}_{C|F}}{\underline{v}_{A|F}} \cdot \overset{\underline{v}_{C|F}}{\underline{v}_{A|F}} + \overset{\underline{v}_{C|F}}{\underline{v}_{A|F}} \cdot m \overset{\underline{0}}{\underline{v}_{CA|F}} + \frac{1}{2} \underline{\omega}_{m|F} \cdot \overset{\underline{H}_{C|F}}{\underline{H}_{A|F}}$$

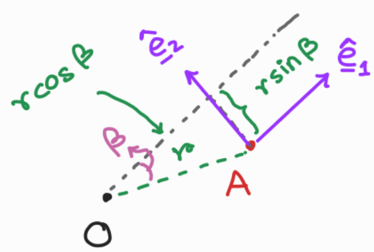
$$= \frac{1}{2} m \underline{v}_{C|F} \cdot \underline{v}_{C|F} + \frac{1}{2} \underline{\omega}_{m|F} \cdot \underline{H}_{C|F}$$

Example: A connecting rod AB (modeled as a thin homogeneous rod) of length ℓ and total mass 'm' is hinged at A, at a distance r from center O of a flywheel that turns with a constant angular velocity $\underline{\omega}_{1|F}$ as shown, in the ground frame 'F', fixed at O.

Determine the kinetic energy of the rod



i) If we consider A as the base point, then we need to use expression B.



$$\underline{v}_{A|F} = \cancel{\underline{v}_{O|F}}^{\,0} + \underline{\omega}_{1|F} \times \underline{r}_{AO}$$

$$= \omega\,\hat{e}_3 \times r\,(\cos\beta\,\hat{e}_1 + \sin\beta\,\hat{e}_2)$$

$$= \omega r\cos\beta\,\hat{e}_2 - \omega r\sin\beta\,\hat{e}_1$$

Angular velocity of rod, $\underline{\omega}_{2|F} = \underline{\omega}_{1|F} + \underline{\omega}_{2|1}$

$$= (\omega + \dot{\beta})\,\hat{e}_3$$

$$\underline{v}_{CA|F} = \underline{\omega}_{2|F} \times \underline{r}_{CA} = (\omega + \dot{\beta})\,\hat{e}_3 \times \frac{\ell}{2}\,\hat{e}_1 = (\omega + \dot{\beta})\,\frac{\ell}{2}\,\hat{e}_2$$

Next, we need to find $[\underline{\underline{I}}^A]$,

We know $[\underline{\underline{I}}^C]$:

$$[\underline{\underline{I}}^C]_{(\hat{e}_1\,\hat{e}_2\,\hat{e}_3)} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{ml^2}{12} & 0 \\ 0 & 0 & \frac{ml^2}{12} \end{bmatrix}$$

Employ parallel-axes thm to get $[\underline{\underline{I}}^A]$

$x_{c_1} = l/2$

$x_{c_2} = x_{c_3} = 0$

$$[\underline{\underline{I}}^A]_{(\hat{e}_1\,\hat{e}_2\,\hat{e}_3)} = \begin{bmatrix} 0 & -m\,x_{c_1}\cancelto{0}{x_{c_2}} & -m\,x_{c_1}\cancelto{0}{x_{c_3}} \\ & \frac{ml^2}{12} + m(x_{c_1}^2 + \cancelto{0}{x_{c_3}^2}) & -m\,x_{c_2}\cancelto{0}{x_{c_3}} \\ \text{SYM} & & \frac{ml^2}{12} + m(x_{c_1}^2 + \cancelto{0}{x_{c_2}^2}) \end{bmatrix}$$

position of C
measured from A
along $\hat{e}_1$

Thus, we get:

$\hat{e}_1$-$\hat{e}_2$-$\hat{e}_3$ coincides
with p-axes at
A for RB

$$[\underline{\underline{I}}^A]_{(\hat{e}_1\,\hat{e}_2\,\hat{e}_3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{ml^2}{3} & 0 \\ 0 & 0 & \frac{ml^2}{3} \end{bmatrix}$$

Now evaluate the expression B

$$\boxed{T_{|F} = \frac{1}{2}\, m\, \underline{v}_{A|F} \cdot \underline{v}_{A|F} + \underline{v}_{A|F} \cdot m\, \underline{v}_{CA|F} + \frac{1}{2}\, \underline{\omega}_{m|F} \cdot \underline{H}_{A|F}} \quad \text{B}$$

$$\frac{1}{2} m \underline{v}_{A|F} \cdot \underline{v}_{A|F} = \frac{1}{2} m \left[(\omega r \cos\beta)^2 + (-\omega r \sin\beta)^2 \right]$$

$$= \frac{1}{2} m \, \omega^2 r^2$$

$$\underline{v}_{A|F} \cdot m \, \underline{v}_{CA|F} = m \left[\underline{v}_{A|F} \right]^T \left[\underline{v}_{CA|F} \right]$$

$$= m \begin{bmatrix} -\omega r \sin\beta & \omega r \cos\beta & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{\omega l}{2} (\dot{\beta} + \omega) \\ 0 \end{bmatrix}$$

$$= m \, \omega \, \frac{l r}{2} (\dot{\beta} + \omega) \cos\beta$$

$$\frac{1}{2} \underline{\omega}_{2|F} \cdot \underline{H}_{A|F} = \frac{1}{2} \left[\underline{\omega}_{2|F} \right]^T \left[H_{A|F} \right]$$

$$= \frac{1}{2} \left[\underline{\omega}_{2|F} \right]^T \left[\underline{\underline{I}}^A \right] \left[\underline{\omega}_{2|F} \right]$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 0 & (\omega + \dot{\beta}) \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{ml^2}{3} & 0 \\ 0 & 0 & \frac{ml^2}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ (\omega + \dot{\beta}) \end{bmatrix}$$

$$= \frac{1}{2} \frac{ml^2}{3} (\omega + \dot{\beta})^2$$

$$T_{|F} = \frac{1}{2} m \left[\omega^2 r^2 + l r \, \omega (\omega + \dot{\beta}) \cos\beta + \frac{l^2}{3} (\omega + \dot{\beta})^2 \right]$$

It took us some amount of work to find the KE of body by taking the base point as A.

2) If we consider the base point as COM C, then we will use expression (A) which is of course a special case of expression (B)

$$T_{|F} = \frac{1}{2} m \underline{v}_{C|F} \cdot \underline{v}_{C|F} + \frac{1}{2} \underline{\omega}_{m|F} \cdot \left(\underline{\underline{I}}^C \underline{\omega}_{m|F} \right)$$

Using velocity-transfer relations to relate velocities at C & A

$$\underline{v}_{C|F} = \underbrace{\underline{v}_{A|F} + \underline{\omega}_{2|F} \times \underline{r}_{CA}}_{\text{known (from prev calc.)}}$$

$$= \omega r \cos\beta \, \hat{\underline{e}}_2 - \omega r \sin\beta \, \hat{\underline{e}}_1 + (\omega + \dot{\beta}) \, \hat{\underline{e}}_3 \times \frac{\ell}{2} \hat{\underline{e}}_1$$

$$= \left[\omega r \cos\beta + (\omega + \dot{\beta}) \frac{\ell}{2} \right] \hat{\underline{e}}_2 - \omega r \sin\beta \, \hat{\underline{e}}_1$$

$$\underline{v}_{C|F} \cdot \underline{v}_{C|F} = \left(\omega r \cos\beta + (\omega + \dot{\beta}) \frac{\ell}{2} \right)^2 + \left(-\omega r \sin\beta \right)^2$$

$$= \omega^2 r^2 \cos^2\beta + 2 \frac{\ell r}{2} \omega (\omega + \dot{\beta}) \cos\beta + (\omega + \dot{\beta})^2 \frac{\ell^2}{4}$$

$$+ \omega^2 r^2 \sin^2\beta$$

$$\underline{\omega}_{m|F} \cdot \underline{H}_{C|F} = \left[\underline{\omega}_{2|F} \right]^T \left[\underline{\underline{I}}^C \right] \left[\underline{\omega}_{2|F} \right]$$

$\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ coincides with p-axes at C

$$= \begin{bmatrix} 0 & 0 & (\omega + \dot{\beta}) \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{m\ell^2}{12} & 0 \\ 0 & 0 & \frac{m\ell^2}{12} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ (\omega + \dot{\beta}) \end{bmatrix}$$

$$= \frac{m\ell^2}{12} (\omega + \dot{\beta})^2$$

$$T_{IF} = \frac{m}{2}\left[\omega^2 r^2 + l r\, \omega(\omega+\dot{\beta})\cos\beta + (\omega+\dot{\beta})^2\left(\frac{l^2}{4}+\frac{l^2}{12}\right)\right]$$

$$= \frac{m}{2}\left[\omega^2 r^2 + l r\, \omega(\omega+\dot{\beta})\cos\beta + \frac{l^2}{3}(\omega+\dot{\beta})^2\right]$$

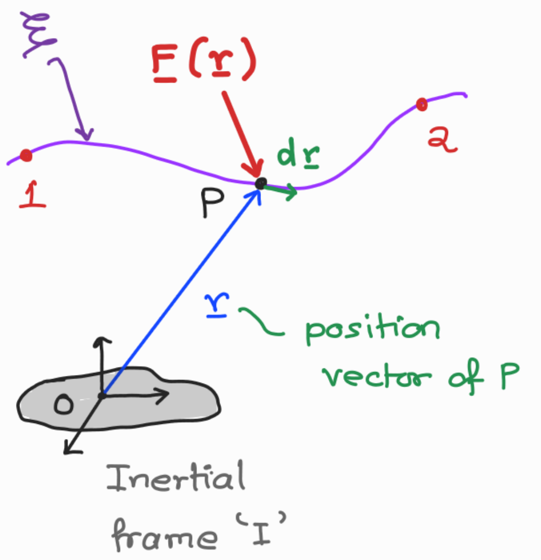
Notice that the total KE of the RB does not depend upon the choice of base point A or COM C, but only depends on the reference frame F. However, depending upon the choice of base point, the computation of KE may became easier!

Work - Energy Principle for a Particle

We will first define work, conservative force and power for a particle and then extend the idea to RBs.

Mechanical work done for a particle

For a particle P in motion along a path ξ due to a force $\underline{F}(\underline{r})$ that varies with particle's position along ξ, the work done by the force $\underline{F}(\underline{r})$ in moving P from point $\underline{r}_1$ to $\underline{r}_2$ is defined by the path integral



$$W = \int_{\xi} \underline{F}(\underline{r}) \cdot d\underline{r} = \int_{\underline{r}_1}^{\underline{r}_2} \underline{F}(\underline{r}) \cdot d\underline{r}$$

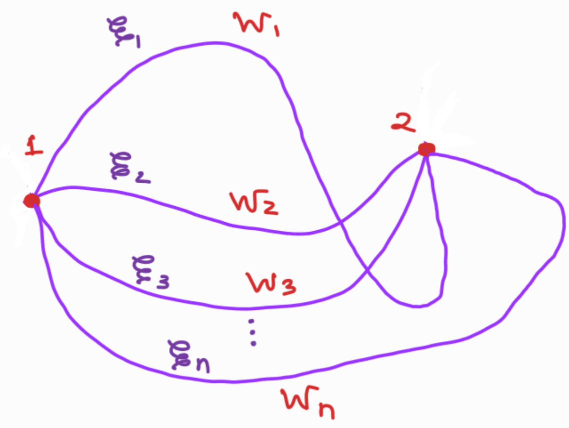
Integration over the path

In Cartesian csys, $W = \int_{x_1, x_2, x_3}^{x_1', x_2', x_3'} (F_1\, dx_1 + F_2\, dx_2 + F_3\, dx_3)$

In path csys, $W = \int_{s_1}^{s_2} F_t\, ds$ ← Only the force comp. tangent to the path does work in moving P

The total work done by force $\underline{F}$ depends on the path traversed by particle P $\rightarrow$ hence path integration is needed

For certain forces, the work done is same for every path joining the same end points, so the work done by these forces depends only on the end point values. A force field $\underline{F}(\underline{r})$ that is independent of path is called a Conservative force



If $\underline{F}(\underline{r})$ is Conservative $\Rightarrow$ $W_1 = W_2 = \cdots = W_n$

(independent of path)

Example: Work done by a constant force

You can show easily that a constant force F_c is conservative

The work done by $\underline{F}_c$ acting between points $\underline{r}_1$ and any other point $\underline{r}_2$ is given by

$$W = \int_{\underline{r}_1}^{\underline{r}_2} \underbrace{\underline{F}_c}_{\text{const.}} \cdot d\underline{r} = \underline{F}_c \cdot \int_{\underline{r}_1}^{\underline{r}_2} d\underline{r} = \underline{F}_c \cdot (\underline{r}_2 - \underline{r}_1)$$

Clearly the work done by the constant force is independent of the path and only depends on end point values

$\Rightarrow$ F_c is conservative force

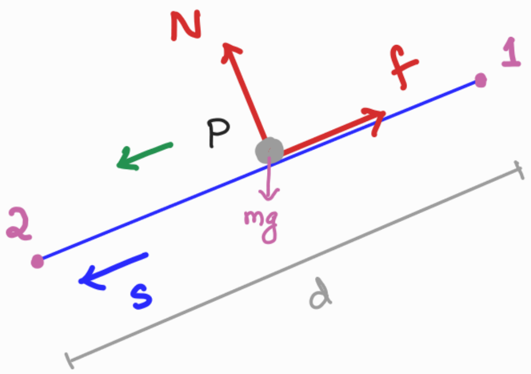
Try Show that linear force field $\underline{F}(\underline{r}) = \underline{F}\,\underline{r}$ is also conservative

A force field $\underline{F}(\underline{r})$ that is not conservative is called a non-conservative force; these are path-dependent forces

e.g. Coulomb friction forces always follow the motion along the specific path of the particle, and they vary with the choice of path between fixed end states.

A Coulomb frictional force even of constant magnitude is NOT a conservative force (How?)

Consider the case of a particle P sliding down an inclined plane



1> the path 's' is a st line given by the incline

2> the normal force remains constant, both in magnitude and direction, along the path 's' i.e. $N(s) = N$

In this case, Coulomb's friction force is a constant force

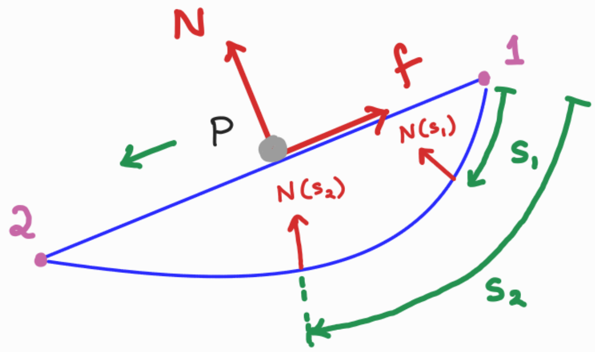
$$f = \mu N$$

Work done by f, $$W_f = \int_0^d f\, ds = \int_0^d \mu N\, ds = \underline{\underline{\mu N d}}$$

For different paths joining the same end states, the distance 'd' along the path connecting these states will be different

$\Rightarrow$ Coulomb frictional force is non-conservative

In general, Coulomb frictional force is not constant and depends on 's'



Note: The mechanical work done by a force $\underline{F}(\underline{r}(t))$ is an explicit function of displacement and implicit function of time.

Mechanical Power for a particle

The total mechanical power is the rate of work done by a force $\underline{F}(\underline{r})$ on moving the particle P

$$\mathcal{P}(t) = \frac{d}{dt}\{W\}\Big|_F = \underline{F}(\underline{r}(t)) \cdot \frac{d\underline{r}}{dt}\Big|_F$$

frame dependent

$$= \underbrace{\underline{F}(\underline{r}(t))}_{\text{frame independent}} \cdot \underbrace{\underline{v}_{P|F}(t)}_{\text{frame dependent}}$$

Work- Energy principle for a particle

Let's now consider the equation of motion of a particle from Newton's 2nd law (using inertial frame for its validity)

$m\underline{a} = \underline{F}$ $\Rightarrow$ $$m \frac{d}{dt}\{\underline{v}_{P|I}\}\Big|_I = \underline{F}$$ ← net resultant force on particle

Take dot product with $\underline{v}_{P|I}$:

$$m \frac{d}{dt}\{\underline{v}_{P|I}\}\Big|_I \cdot \underline{v}_{P|I} = \underline{F} \cdot \underline{v}_{P|I}$$

$$\Rightarrow \frac{m}{2} \frac{d}{dt}\{\underline{v}_{P|I} \cdot \underline{v}_{P|I}\}\Big|_I = \underline{F} \cdot \underline{v}_{P|I}$$

$$\Rightarrow \frac{d}{dt}\left\{\frac{1}{2} m\, \underline{v}_{P|I} \cdot \underline{v}_{P|I}\right\}\Big|_I = \underline{F} \cdot \underline{v}_{P|I}$$

$$\Rightarrow \frac{d}{dt}\{\underbrace{T|_I}_{\text{Mech. KE}}\}\Big|_I = \mathcal{P}$$ ← Mech. power

Integrating the above over the path ξ traversed by the particle from $\underline{r}_1 = \underline{r}(t_1)$ to $\underline{r}_2 = \underline{r}(t_2)$, we obtain the

Work- Energy principle :

$$\underbrace{\Delta T}_{T(t_2) - T(t_1)} = W_{1-2}$$

(dropping the subscript for inertial frame)

It states that the work done by the force $\underline{F}(\underline{r})$ acting on a particle over its path ξ from time t_1 to time t_2 is equal to the change in the kinetic energy of that particle in that time

In terms of mechanical power, the power expended by the force is equal to the time rate of change of kinetic energy of the particle

$$P = \left.\frac{dW}{dt}\right|_I = \frac{d}{dt}\left\{ T|_I \right\}\Big|_I$$

Advantages of Work-Energy Principle

A major advantage of the work-energy principle is that it avoids the necessity of computing accelerations and leads directly to the velocity changes as functions of the forces which do the work. Furthermore, the principle only involves those forces which do work (i.e. applied forces) and not those that don't do work (e.g. reaction forces / couples).

Also, it is a scalar equation (instead of 3 scalar equations)