Recap

Euler's 2nd axiom in terms of inertia tensor

$$\underline{\underline{I}}^A \dot{\underline{\omega}}_{m|I} + \underline{\omega}_{m|I} \times (\underline{\underline{I}}^A \underline{\omega}_{m|I}) = \underline{M}_A$$

Euler's axioms for general motion of an RB

net resultant force

1st : $\underline{F}_R = m\, a_{c|I}$ Inertia frame COM

net moment abt A

2nd : $\underline{M}_A = \underline{\underline{I}}^A \dot{\underline{\omega}}_{m|I} + \underline{\omega}_{m|I} \times (\underline{\underline{I}}^A \underline{\omega}_{m|I})$

valid pt

$(\dot{\underline{H}}_{A|I} = \underline{M}_A)$

e.g. $A \equiv COM$

In the last class, we have seen two simplified situations of Euler's 2nd axiom

1) The working csys $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ coincides with p-axes

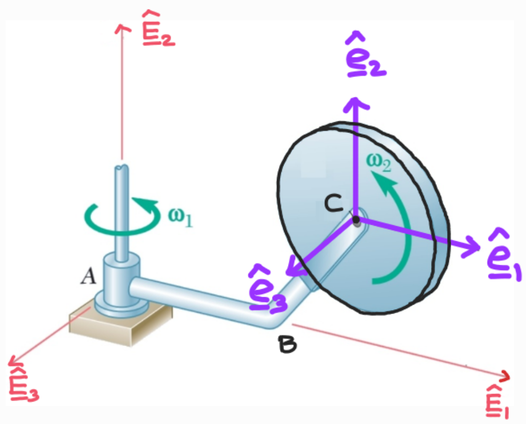
$$[\underline{\underline{I}}^A]_{(\hat{\underline{e}}_1 \hat{\underline{e}}_2 \hat{\underline{e}}_3)} = \begin{bmatrix} I_{11}^A & 0 & 0 \\ 0 & I_{22}^A & 0 \\ 0 & 0 & I_{33}^A \end{bmatrix}, \quad \dot{\underline{\omega}}_{m|I} = \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix}, \quad \underline{\omega}_{m|I} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$\underline{M}_A = \left(I_{11}^A \dot{\omega}_1 - (I_{22}^A - I_{33}^A)\, \omega_2 \omega_3\right) \hat{\underline{e}}_1$$

$$+ \left(I_{22}^A \dot{\omega}_2 - (I_{33}^A - I_{11}^A)\, \omega_3 \omega_1\right) \hat{\underline{e}}_2$$

$$+ \left(I_{33}^A \dot{\omega}_3 - (I_{11}^A - I_{22}^A)\,\omega_1 \omega_2 \right) \hat{e}_3$$

or,



$$M_{A,1} = I_{11}^A \dot{\omega}_1 - (I_{22}^A - I_{33}^A)\,\omega_2 \omega_3$$

$$M_{A,2} = I_{22}^A \dot{\omega}_2 - (I_{33}^A - I_{11}^A)\,\omega_3 \omega_1$$

$$M_{A,3} = I_{33}^A \dot{\omega}_3 - (I_{11}^A - I_{22}^A)\,\omega_1 \omega_2$$

2) RB rotating about a body-fixed axis (say $\hat{e}_3$)

$$[\underline{\underline{I}}^A]_{(\hat{e}_1 \hat{e}_2 \hat{e}_3)} = \begin{bmatrix} I_{11}^A & I_{12}^A & I_{13}^A \\ & I_{22}^A & I_{23}^A \\ \text{sym} & & I_{33}^A \end{bmatrix}, \quad \dot{\underline{\omega}}_{m|I} = \begin{bmatrix} 0 \\ 0 \\ \dot{\omega} \end{bmatrix}, \quad \underline{\omega}_{m|I} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$

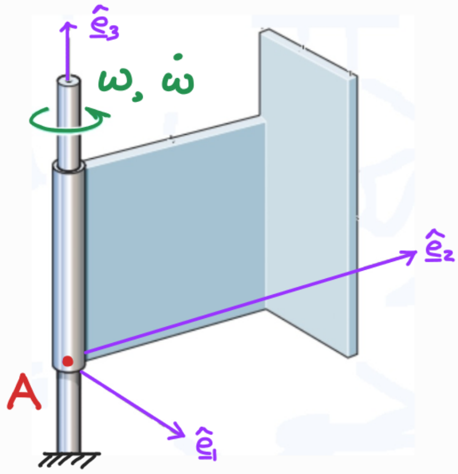
$$\underline{M}_A = \left(I_{13}^A \dot{\omega} - I_{23}^A \omega^2 \right) \hat{e}_1 + \left(I_{23}^A \dot{\omega} + I_{13} \omega^2 \right) \hat{e}_2 + I_{33}^A \dot{\omega}\, \hat{e}_3$$

or

$$M_{A,1} = I_{13}^A \dot{\omega} - I_{23}^A \omega^2$$

$$M_{A,2} = I_{23}^A \dot{\omega} + I_{13} \omega^2$$

$$M_{A,3} = I_{33}^A \dot{\omega}$$



Application 3 : General plane (2D) motion

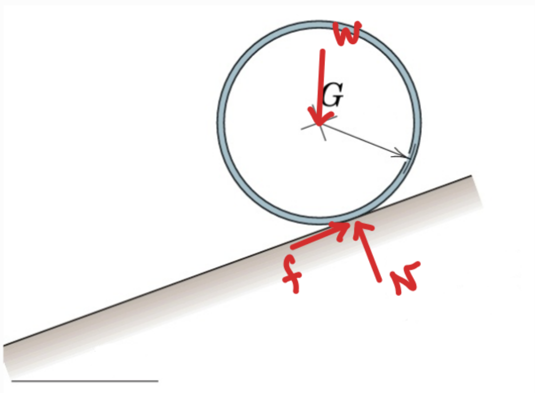
Plane motion of an RB implies a motion for which every material point of the RB moves parallel to a specified plane

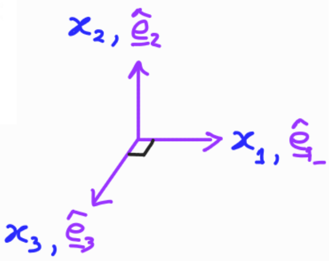
cannot be massless

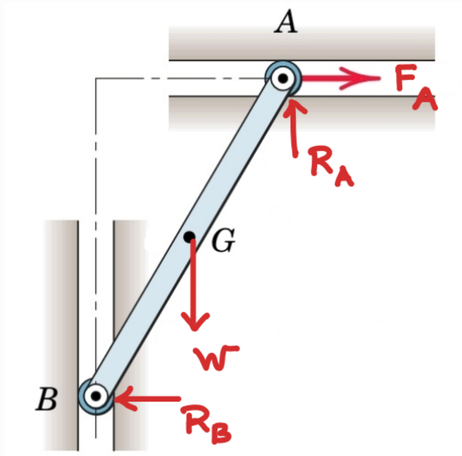
[say $\hat{e}_1 - \hat{e}_2$ plane (or $x_1 - x_2$ plane)]

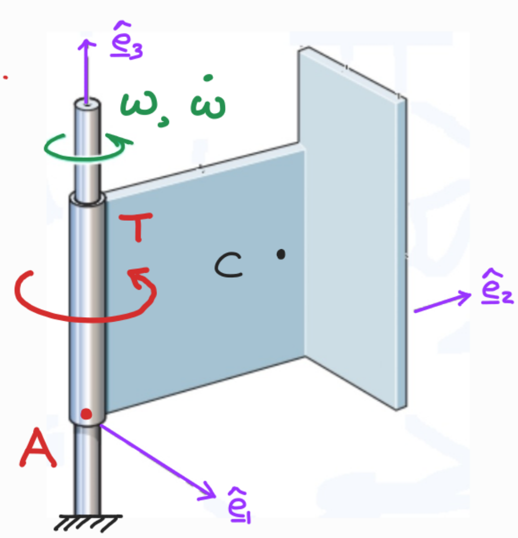
Net external force (appl. and reactions) must lie in $x_1 - x_2$ plane and all external moments (appl. & reactions) are $\perp$ to the $x_1 - x_2$ plane! → **Coplanar force system**

Examples:

① 



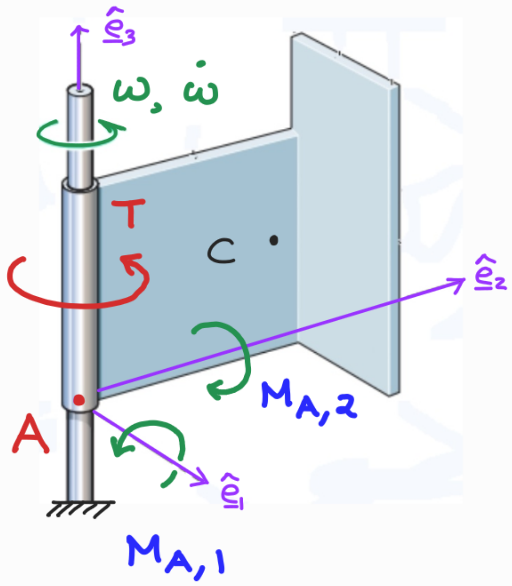
② 

③ e.g. 

← For this example, it can be considered 2D plane motion ONLY IF the reaction forces out of the $x_1 - x_2$ plane are zero and the reaction couples along $\hat{e}_1$ and $\hat{e}_2$ dir's are zero.

Let's use this example and derive simplified Euler's equations for plane motion of an RB!

Recall the simplified Euler's 2nd equation for the rotation of RB about a fixed axis (say $\hat{e}_3$-axis)



$$\underline{M}_A = \underbrace{(I_{13}^A \dot{\omega} - I_{23} \omega^2)}_{M_{A,1}} \hat{\underline{e}}_1 + \underbrace{(I_{23}^A \dot{\omega} + I_{13}^A \omega^2)}_{M_{A,2}} \hat{\underline{e}}_2 + \underbrace{I_{33}^A \dot{\omega}}_{M_{A,3}} \hat{\underline{e}}_3$$

Support reaction couples

$M_{A,3} = T$ (Applied torque)

For plane motion of an RB in $x_1 - x_2$ plane $\rightarrow$ all external moments about $\hat{\underline{e}}_1$ and $\hat{\underline{e}}_2$ must vanish

$\Downarrow$

Reaction couples

$M_{A,1} = 0$ & $M_{A,2} = 0$

$\Downarrow$

$I_{13}^A = I_{23}^A = 0$

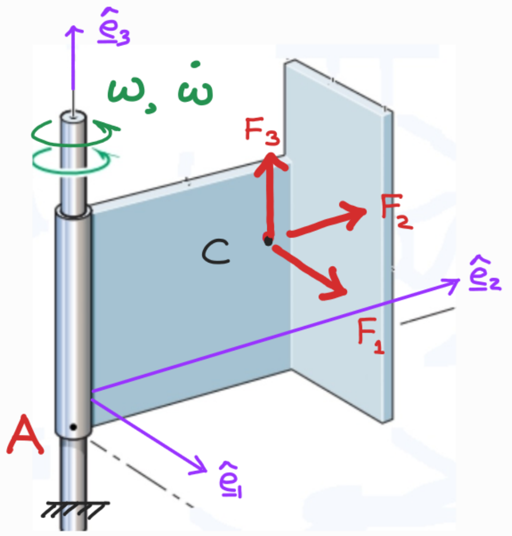
(for arbitrary $\omega, \dot{\omega}$)

$\Leftarrow$

The axis of rotation $\hat{\underline{e}}_3$ must coincide with a p-axis of RB

$\hookrightarrow$ **point A = C** (since p-axes always pass thru COM of an RB)

Applying Euler's 1st equation:



$$\underline{F}_R = m\,\underline{a}_{C|I} = a_{c,1}\underline{\hat{e}}_1 + a_{c,2}\underline{\hat{e}}_2 + a_{c,3}\underline{\hat{e}}_3$$

$$\Rightarrow \begin{aligned} F_1\,\underline{\hat{e}}_1 &+ & m\,a_{c,1}\,\underline{\hat{e}}_1 \\ F_2\,\underline{\hat{e}}_2 &+ \quad = & m\,a_{c,2}\,\underline{\hat{e}}_2 \\ F_3\,\underline{\hat{e}}_3 & & m\,a_{c,3}\,\underline{\hat{e}}_3 \end{aligned}$$

For 2D plane motion of RB, the net ext. force out of the x_1-x_2 plane must be zero.

$$\Downarrow$$

$$F_3 = 0$$

$$\Downarrow$$

$$a_{c,3} = 0$$

$$\Downarrow$$

The component of linear acceleration out of the x_1-x_2 plane must be zero

Plane 2D motion of an RB can be considered as a special case of rotation of RB about body-fixed axis with a difference

For plane (2D) motion, only the $\hat{e}_3$-axis must remain fixed relative to the body and the inertial frame I

(A) $\Rightarrow \frac{d}{dt}\{\hat{\underline{e}}_3\}\Big|_m = \frac{d}{dt}\{\hat{\underline{e}}_3\}\Big|_I = 0$ for plane motion

BUT for rotation about body-fixed axis, all $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ axes must remain fixed to the body in addition to $\hat{\underline{e}}_3$ axis remaining fixed relative to body and inertial frame I

(A) + (B) $$\left.\begin{aligned}\frac{d}{dt}\{\hat{\underline{e}}_1\}\Big|_m = \frac{d}{dt}\{\hat{\underline{e}}_2\}\Big|_m = \frac{d}{dt}\{\hat{\underline{e}}_3\}\Big|_m = 0\end{aligned}\right\} \text{(B)}$$

Plane motion in $x_1 - x_2$ plane

1) $\underline{\omega}_{m|I} = \omega\, \hat{\underline{e}}_3$, $\dot{\underline{\omega}}_{m|I} = \dot{\omega}\, \hat{\underline{e}}_3$, $\frac{d}{dt}\hat{\underline{e}}_3\Big|_I = \underline{0}$

2) $\underline{e}_3$-axis must coincide with a p-axis at the COM C

3) the $\underbrace{\text{out-of-plane linear acceleration component}}_{a_{c,3}}$ is zero

Therefore, simplified Euler's equations for plane motion:

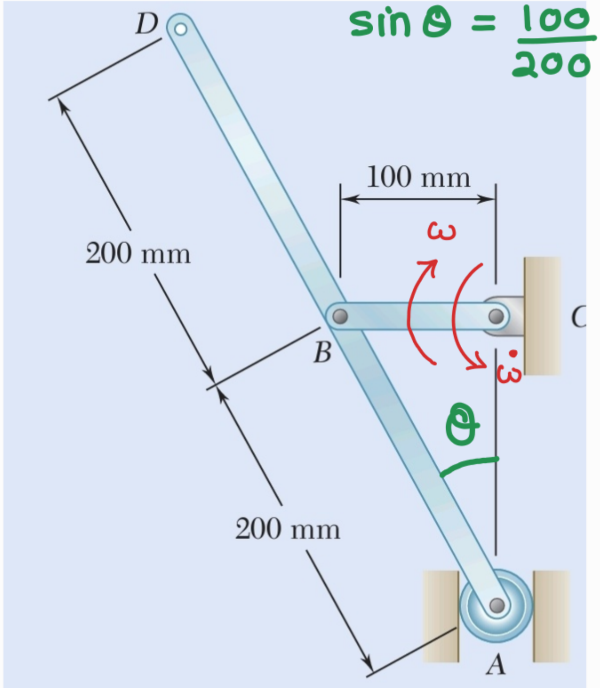
$$\boxed{\begin{aligned} F_1 &= m\, a_{c,1} \\ F_2 &= m\, a_{c,2} \\ M_{c,3} &= I_{33}^{c}\, \dot{\omega} \end{aligned}}$$

Example

P16.124

(Beer & Johnston)

The 4-kg uniform rod *ABD* is attached to the crank *BC* and is fitted with a small wheel that can roll without friction along a vertical slot. Knowing that at the instant shown crank *BC* rotates with an angular velocity of 6 rad/s clockwise and an angular acceleration of 15 rad/s^2 counterclockwise, determine the reaction at *A*. and B (exerted on rod AD)



Solution : You are asked to find reaction forces at the instant shown

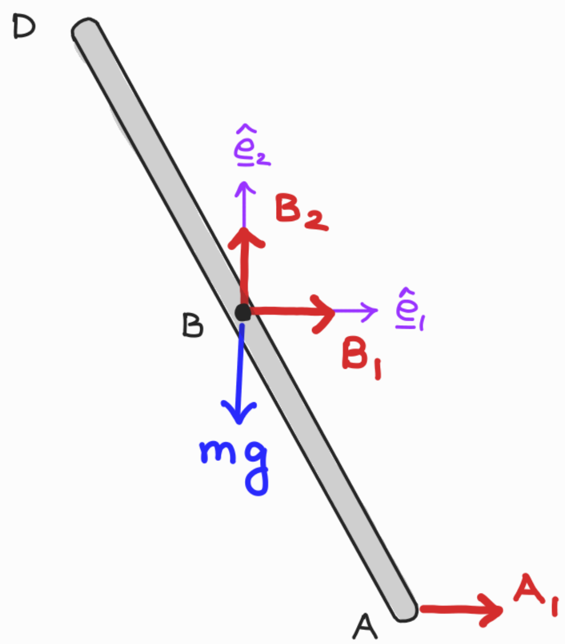
- Want to determine reaction forces at A and B.

- Both points lie on rod ABD so start by choosing ABD as the RB of interest

- For plane motion (in x_1-x_2 plane), lets fix our csys $\hat{\underline{e}}_1$ - $\hat{\underline{e}}_2$ - $\hat{\underline{e}}_3$ at the COM B of the rod, with $\hat{\underline{e}}_3$ as p-axis

- Draw FBD
  - Pin joint reactions → horizontal B_1, vertical B_2
  - Roller support reactions → horizontal A_1
  - Weight due to gravity mg

FBD



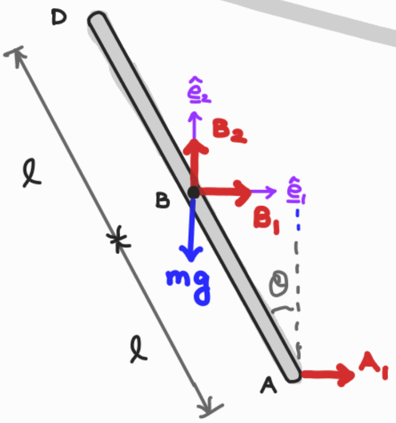
– Applying the simplified Euler's equations of motion for plane (2D) motion:

$$\overset{+}{\rightarrow} \Sigma F_{R,1} = m_{AB}\, a_{B,1} \Rightarrow \boxed{A_1} + \boxed{B_1} = m\, \boxed{a_{B,1}} \quad \text{— (1)}$$

$$+\uparrow \Sigma F_{R,2} = m_{AB}\, a_{B,2} \Rightarrow \boxed{B_2} - mg = m\, \boxed{a_{B,2}} \quad \text{— (2)}$$

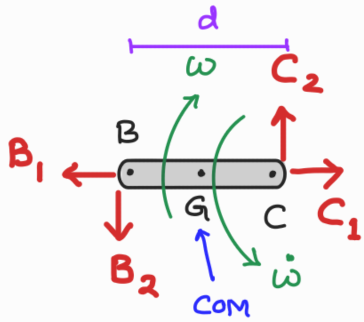
$$\circlearrowleft + \Sigma M_{B,3} = I^{B}_{33}\, \dot{\omega}_{AB} \Rightarrow A_1\,(0.1) = \frac{4}{75}\, \boxed{\dot{\omega}_{AB}} \quad \text{— (3)}$$

$$\frac{m\,(2l)^2}{12} = \frac{4\,(2\times 0.2)^2}{12} = \frac{4}{75}$$



$$l\cos\theta = \frac{0.2}{2} = 0.1$$

In the three equations (1), (2), & (3), we have got SIX unknowns: A_1, B_1, B_2, $a_{B,1}$, $a_{B,2}$, $\dot{\omega}_{AB}$. Need more equations for solving the unknowns.



OPTIONS:

→ Choose rod BC as another RB but that will introduce 4 more unknowns and m_{BC} not given!

AND/OR

→ Turn to kinematics for more equations

$$\boxed{C_2} - B_2 = m_{BC}\, \boxed{a_{G,2}}$$

$$\boxed{C_1} - B_1 = m_{BC}\, \boxed{a_{G,1}}$$

$$\checkmark\, \boxed{B_2}\, d = I^{G}_{33}\, \dot{\omega}_{BC}$$

- Kinematics: Since $\underline{\omega}_{BC}$ and $\dot{\underline{\omega}}_{BC}$ are given, lets find

$$\underline{a}_{B|I} = \underline{a}_{C|I} + \dot{\underline{\omega}}_{BC} \times \underline{r}_{BC} + \underline{\omega}_{BC} \times (\underline{\omega}_{BC} \times \underline{r}_{BC})$$

C is fixed in 'I'

$$= 15\,\hat{\underline{e}}_3 \times \underbrace{(-0.1\,\hat{\underline{e}}_1)}_{\underline{r}_{BC}} + (-6\,\hat{\underline{e}}_3) \times \{(-6\,\hat{\underline{e}}_3) \times \underbrace{(-0.1\,\hat{\underline{e}}_1)}_{\underline{r}_{BC}}\}$$

$$\underline{a}_{B|I} = \underbrace{-1.5}_{a_{B,2}}\,\hat{\underline{e}}_2 + \underbrace{3.6}_{a_{B,1}}\,\hat{\underline{e}}_1 \Rightarrow \begin{array}{l} a_{B,1} = 3.6 \ m/s^2 \\ a_{B,2} = -1.5 \ m/s^2 \end{array}$$

$\underline{a}_{B|I}$ known ✓

Still 4 unknowns remaining and 3 equations only!

Let's use more kinematics, this time using known acc. at point A: $a_{A,1} = 0$, to determine $\underline{a}_{B|I}$!

$$\underline{a}_{B|I} = \underline{a}_{A|I} + \dot{\underline{\omega}}_{AB} \times \underline{r}_{BA} + \underline{\omega}_{AB} \times (\underline{\omega}_{AB} \times \underline{r}_{BA})$$

$$= a_{A,2}\,\hat{\underline{e}}_2 + \underbrace{(\dot{\omega}_{AB}\,\hat{\underline{e}}_3) \times (-0.1\,\hat{\underline{e}}_1 + 0.2\sin\theta\,\hat{\underline{e}}_2)}_{a}$$

$$+ \underbrace{(\omega_{AB}\,\hat{\underline{e}}_3) \times \{(\omega_{AB}\,\hat{\underline{e}}_3) \times (-0.1\,\hat{\underline{e}}_1 + 0.2\sin\theta\,\hat{\underline{e}}_2)\}}_{b}$$

$\cos\theta = \frac{\sqrt{3}}{2}$

a

$$(\dot{\omega}_{AB}\,\hat{\underline{e}}_3) \times (-0.1\,\hat{\underline{e}}_1 + 0.2\cos\theta\,\hat{\underline{e}}_2) = -0.2\cos\theta\,\dot{\omega}_{AB}\,\hat{\underline{e}}_1 - 0.1\,\dot{\omega}_{AB}\,\hat{\underline{e}}_2$$

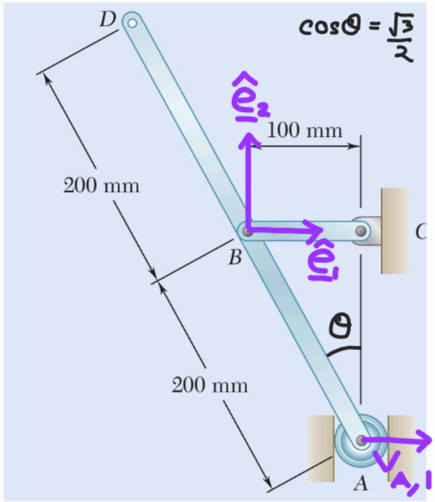
b

$$(\omega_{AB}\,\hat{\underline{e}}_3) \times \{(\omega_{AB}\,\hat{\underline{e}}_3) \times (-0.1\,\hat{\underline{e}}_1 + 0.2\cos\theta\,\hat{\underline{e}}_2)\} = 0.1\,\omega_{AB}^2\,\hat{\underline{e}}_1 - 0.2\cos\theta\,\omega_{AB}^2\,\hat{\underline{e}}_2$$

$$\underline{a_{B,1}}\,\hat{\underline{e}}_1 + a_{B,2}\,\hat{\underline{e}}_2 = \underline{\left(-0.2\cos\theta\,\dot{\omega}_{AB} + 0.1\,\omega_{AB}^2\right)}\,\hat{\underline{e}}_1 + \left(a_{A,2} - 0.1\,\dot{\omega}_{AB} - 0.2\cos\theta\,\omega_{AB}^2\right)\hat{\underline{e}}_2$$

ω_{AB} (?) Calculate $\underline{\omega}_{AB}$ by using velocity transfer rels. (Kinematics again)

$$\underline{v}_{B|I} = \underline{v}_{A|I} + \underline{\omega}_{AB} \times \underline{r}_{BA}$$



Looking at $\hat{e}_1$ component only (since $v_{A,1} = 0$)

$$\Rightarrow v_{B,1}\,\hat{\underline{e}}_1 = v_{A,1}\,\hat{\underline{e}}_1 + \omega_{AB}\,\hat{\underline{e}}_3 \times \left(-0.1\,\hat{\underline{e}}_1 + 0.2\cos\theta\,\hat{\underline{e}}_2\right)$$

$$\Rightarrow v_{B,1} = \cancelto{0}{v_{A,1}} - 0.2\cos\theta\,\omega_{AB}$$

$$\underline{v}_{B|I} = \underline{v}_{C|I} + \underline{\omega}_{BC} \times \underline{r}_{BC}$$

$$\Rightarrow v_{B,1} = v_{C,1} + \underbrace{\left(-6\,\hat{\underline{e}}_3\right) \times \left(-0.1\,\hat{\underline{e}}_1\right)}_{\text{has no component along } \hat{\underline{e}}_1}$$

$$\Rightarrow v_{B,1} = \cancelto{0}{v_{C,1}} \quad (\because \text{ pt C is fixed to I})$$

$$\Rightarrow \underline{v}_{B,1} = 0$$

So we get $\omega_{AB} = 0$

then, $$\dot{\omega}_{AB} = \frac{a_{B,1}}{-0.2\cos\theta} = \frac{3.6}{-0.2\,\frac{\sqrt{3}}{2}} = 12\sqrt{3}\ \text{rad/s}^2$$

Now solve the 3 equations and get 3 reactions:

$$A_1 + B_1 = m\, a_{B,1} \quad \text{— (1)}$$

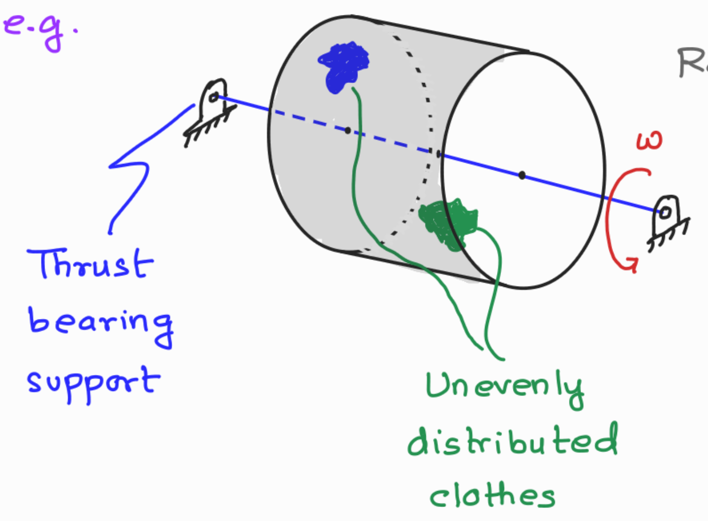
$$B_2 - mg = m\, a_{B,2} \quad \text{— (2)}$$

$$A_1 (0.1) = \frac{4}{75}\, \dot{\omega}_{AB} \quad \text{— (3)}$$

Application 4 : Balanced motion of a rotor

Rotors are RBs that rotates very fast about an axis such as rotors in washing machines, crankshafts in engines.

e.g.



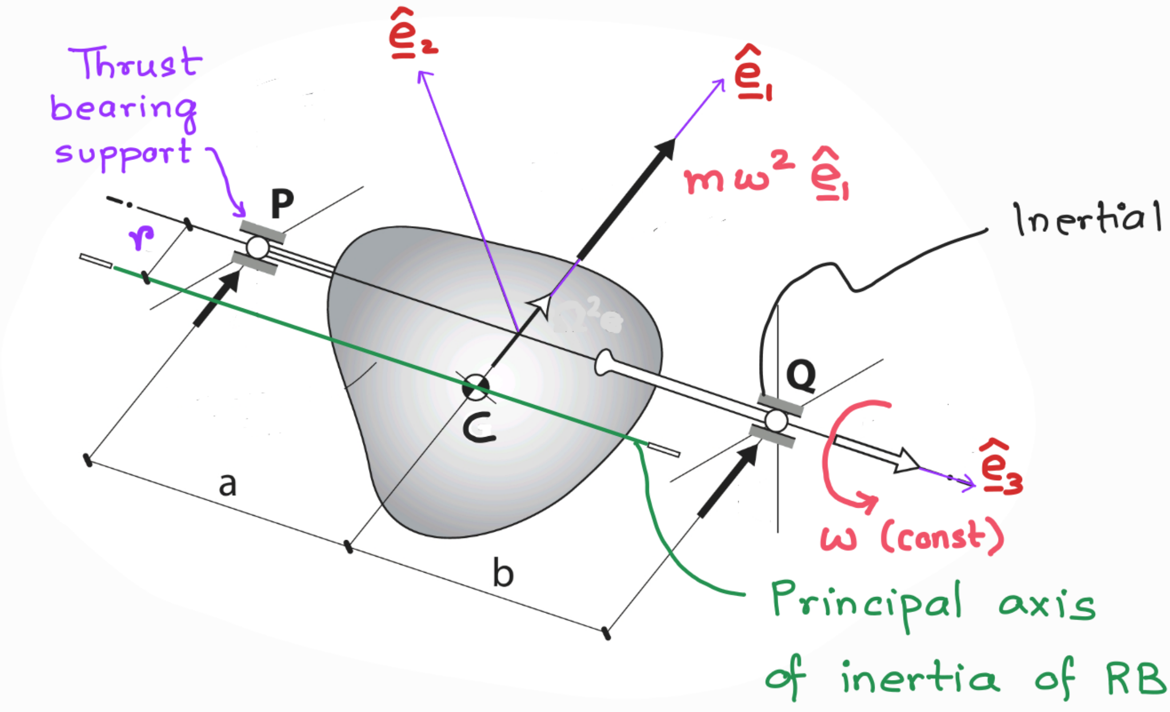
Rotating drum of a front-loaded washing machine

Rotor imbalance (in say washing machine)

- When drum spins, the clothes are unevenly distributed
- The COM shifts away from the axis of rotation
- This causes an unbalanced centrifugal force that increases with rotational speed
- This unbalanced centrifugal force causes the dynamic bearing forces and couples about the rotor axis to alternate periodically, with period = one revolution of the rotor.
- These dynamic bearing forces and couples cause vibrations

of the bearings which are transmitted to the thrust bearing supports, and can cause unpleasant noise and even failure of the machine.

Hence, we need to prevent these unbalanced forces and couples.



Rotor imbalance is caused due to unsymmetric mass dist.

→ So modifying the rotor mass dist is called ROTOR BALANCING

Balanced motion of rotor

⇓

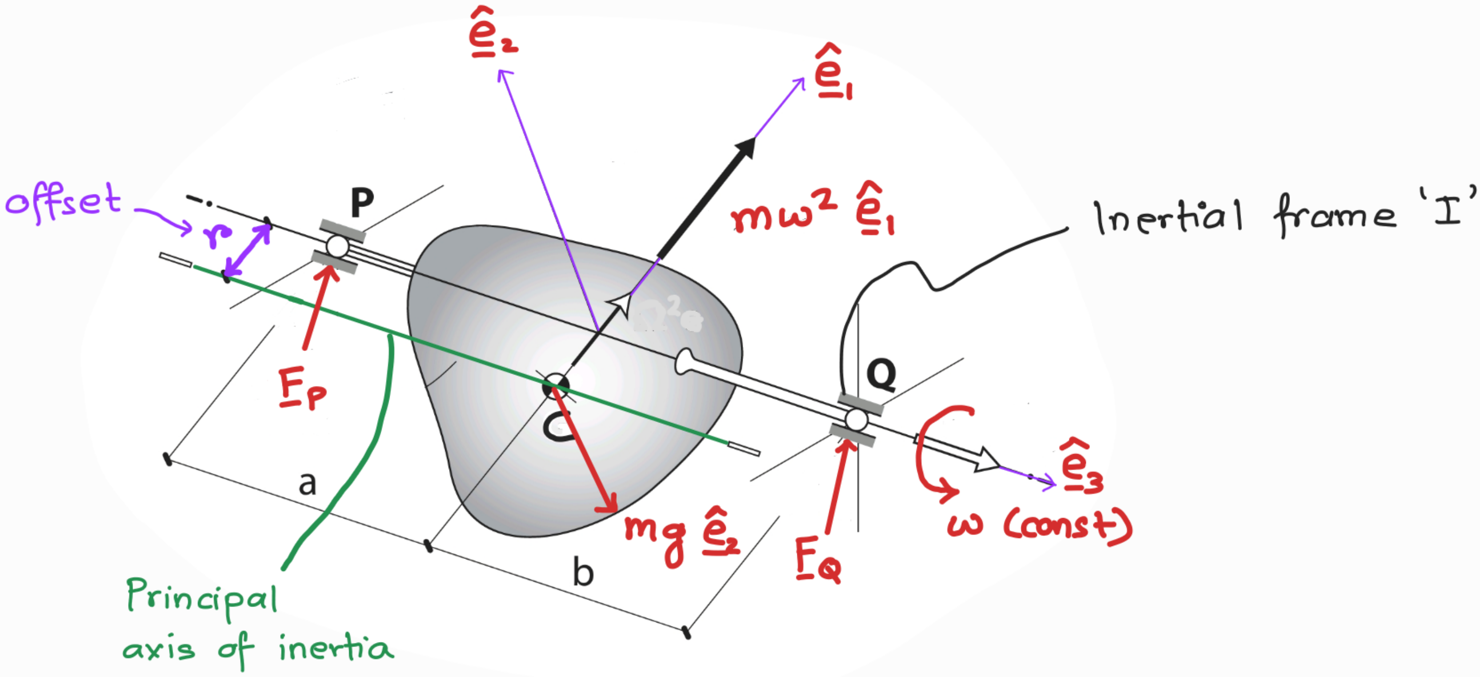
means

1) Reaction force system is independent of motion of rotor.

2) Reaction forces only balance the rotor's weight

3) All reaction moments of couples are zero

Consider a principal axis of inertia of the rotor to be parallel to the rotating axis ($\hat{e}_3$ in this case)



- Rotor (RB) with mass m rotating with constant $\omega\,\hat{e}_3$

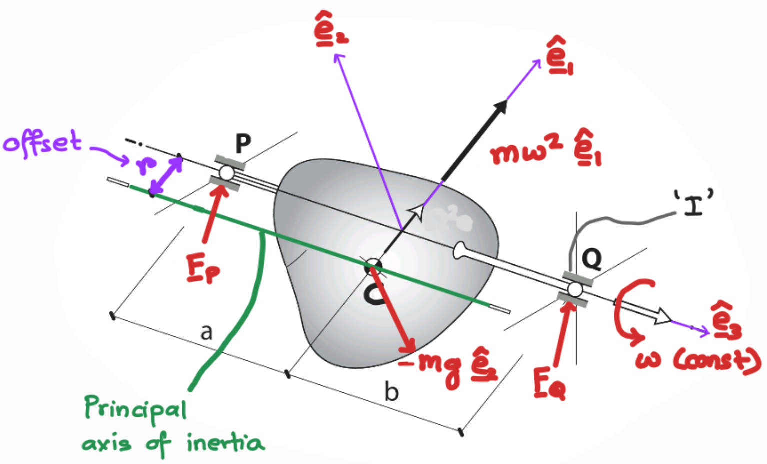
- P, Q are thrust bearing supports ($P_3 = 0$, $Q_3 = 0$)

- COM C of rotor has offset 'r' from axis of rotation

- If a rotation axis $\hat{e}_3$ is parallel to a principal axis of inertia through C, then the acceleration of COM C at the time instant shown, $\underline{a}_{C|I} = \omega^2 r\,\hat{e}_1$. Note the direction changes in the next time instant.

Using Euler's 1st eqn for translational motion:

$$\underline{F}_R = m\,\underline{a}_{C|I}$$



$$\Rightarrow \underbrace{\underline{F}_P + \underline{F}_Q}_{\text{reaction forces}} - mg\,\hat{\underline{e}}_2 = m\omega^2 r\,\hat{\underline{e}}_1$$

For the reaction forces to be independent of the rotational motion:

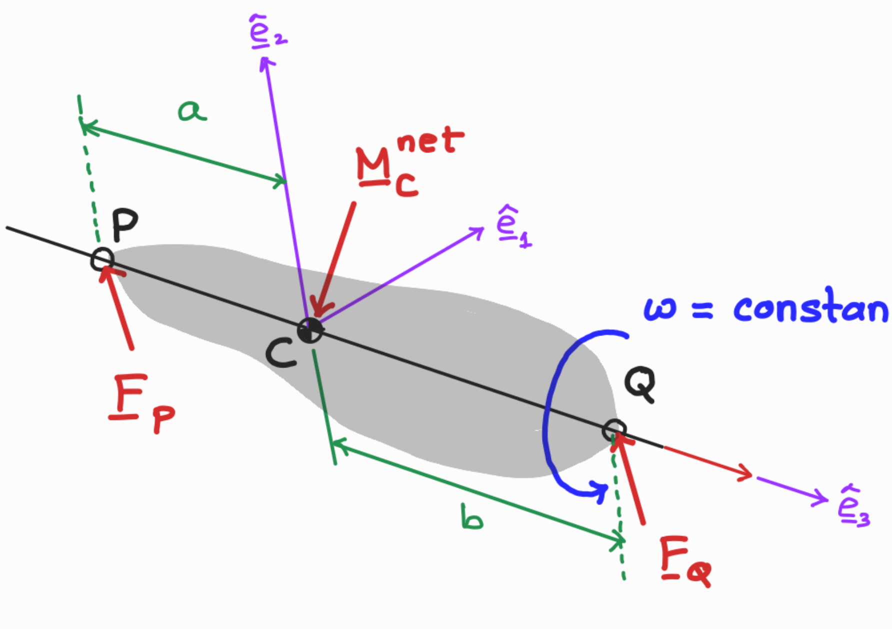
The COM C must lie on the axis of rotation (i.e. $r = 0$)

$$\Rightarrow \underbrace{\underline{F}_P + \underline{F}_Q}_{} = mg\,\hat{\underline{e}}_2 \quad \text{(A)} \quad \text{(Necessary condition)}$$

has components along $\hat{\underline{e}}_1$ and $\hat{\underline{e}}_2$
since the support allows movement in $\hat{\underline{e}}_3$

The reaction forces balances only the weight of rotor

Place COM C of the rotor on the rotating axis $\hat{\underline{e}}_3$ ($r=0$),



$$\underline{F}_P = F_{P,1}\,\hat{\underline{e}}_1 + F_{P,2}\,\hat{\underline{e}}_2$$

$$\underline{F}_Q = F_{Q,1}\,\hat{\underline{e}}_1 + F_{Q,2}\,\hat{\underline{e}}_2$$

and use Euler's 2nd equation for rotational motion.

We now have Case 2 (rotation about a fixed RB axis)

ω = constant, $\dot{\omega} = 0$,

$$\left.\begin{aligned} M_{c,1} &= I_{13}^{c}\,\cancel{\dot{\omega}}^{\,0} - I_{23}^{c}\,\omega^2 \\ M_{c,2} &= I_{23}^{c}\,\cancel{\dot{\omega}}^{\,0} + I_{13}^{c}\,\omega^2 \\ M_{c,3} &= I_{33}^{c}\,\cancel{\dot{\omega}}^{\,0} = 0 \end{aligned}\right\}\ \text{holds!}$$

Since the reaction couples are also independent of rotational motion $\Rightarrow$ $\underbrace{M_{c,1} = M_{c,2} = 0}$ (else they will depend on ω^2)

$\Downarrow$

is automatically satisfied if rotating axis ($\hat{e}_3$) coincides with a principal axis of the rotor as I_{13}^{c} and I_{23}^{c} will vanish — B (Sufficient condn)

Thus, for a rotor to be "balanced", the necessary & sufficient conditions are:

1) COM of rotor must lie on the axis of rotation (say $\hat{e}_3$)

2) The axis of rotation must coincide with a principal axis of the rotor at COM.