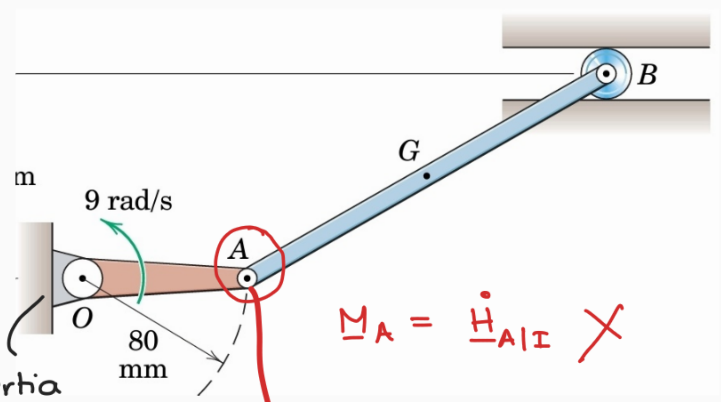
We have discussed about Euler's 2nd axiom at a stretch now, where after embedding the csys to the frame 'm' of the RB, we obtained a usable form of Euler's 2nd axiom for an RB

$$\dot{\underline{H}}_{A|I} = \underline{\underline{I}}^A \, \dot{\underline{\omega}}_{m|I} + \underline{\omega}_{m|I} \times \left(\underline{\underline{I}}^A \, \underline{\omega}_{m|I} \right) = \underline{M}_A \quad — (*)$$

Thus, bear in mind, that in applications of Euler's axiom for motion of an RB in an inertial frame 'I', all quantities in (*) must be calculated in an RB-fixed frame using a csys fixed to the RB.

In most common cases, point A must be either fixed (or in uniform motion $\underline{a}_{A|I} = \underline{0}$) w.r.t inertial frame 'I', or at the COM of the RB. Other case is $\underline{a}_{A|I}$ directed through COM.

Note: The hinge point A in the example, violates these conditions for 'A' to be a valid point for satisfying (*), hence cannot be used to determine $\underline{M}_A$



Inertia frame 'I'

$\underline{M}_A = \dot{\underline{H}}_{A|I}$ ✗

accelerating pt.

Euler's Equations for Dynamics of RBs

The two Euler's axioms provide a system of differential equations for the general motion of an RB given by the two vector ODEs

COM

$$\underline{F}_R = m\,\underline{a}_{C|I} \quad \text{—} \quad \text{I}$$

$$\underline{M}_A = \underline{\underline{I}}^A\,\dot{\underline{\omega}}_{m|I} + \underline{\omega}_{m|I} \times \left(\underline{\underline{I}}^A\,\underline{\omega}_{m|I}\right) \quad \text{—} \quad \text{II}$$

This 2nd equation is a coupled system of three nonlinear ODE for rotational motion of an RB.

Note: A is a point chosen such that $\dot{\underline{H}}_{A|I} = \underline{M}_A$

Pure translation : $\Rightarrow \underline{\omega}_{m|I} = \underline{0}$ and $\dot{\underline{\omega}}_{m|I} = \underline{0} \Rightarrow \underline{M}_A = \underline{0}$

$\underline{F}_R = m\,\underline{a}_{C|I}$ ⟶ determines translation motion of COM

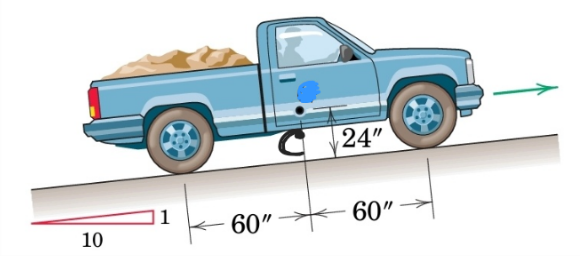
$\underline{M}_A = \underline{0}$ ⟶ provides relations among forces that act on the RB

Pure translation
- Rectilinear — RB translating in st. line
- Curvilinear — RB translating along a curved line

Lets consider an example of rectilinear translation

Ex

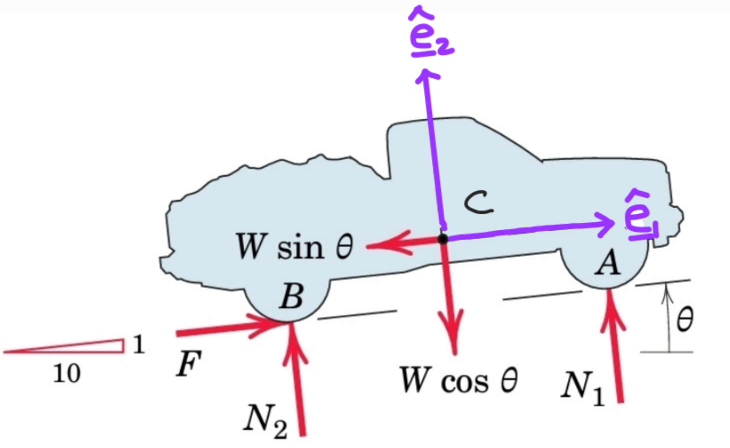
The pickup truck weighs 3220 lb and reaches a speed of 44 ft/s from rest in a distance of 200 ft up the 10-percent incline with constant acceleration. Calculate the normal force under each pair of wheels and the friction force under the rear driving wheels. The kinetic coefficient of friction between the tires and the road is known to be at least 0.80.



Assume: 1) Mass of the wheels is negligible

2) Truck can be simulated as a single RB

FBD of truck



- Fix a csys $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$ to the RB with convenient directions (Choose $\hat{e}_1$ along the st. line path $\Rightarrow$ linear acc. component along $\hat{e}_2$ vanishes)

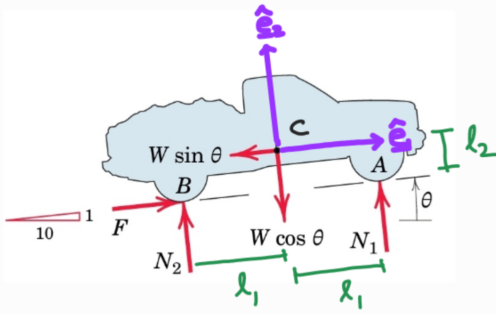
$$\underline{a}_{C|I} = a_1 \hat{\underline{e}}_1 + \cancel{a_2}^{\,0} \hat{\underline{e}}_2 + \cancel{a_3}^{\,0} \hat{\underline{e}}_3$$

Similarly, for straight line motion: $\underline{v}_{C|I} = v_1 \hat{\underline{e}}_1$

Using the relation: $v_1^2 = 2 a_1 s_1$

$$\Rightarrow a_1 = \frac{(44)^2}{2(200)} = 4.84 \text{ ft/s}^2$$

Equivalent force system at C



$$\underline{F}_R = (F - W\sin\theta)\,\underline{\hat{e}}_1 + (N_1 + N_2 - W\cos\theta)\,\underline{\hat{e}}_2$$

$$\underline{M}_C = \left[(N_1 - N_2)\,\ell_1 + F\ell_2\right]\underline{\hat{e}}_3$$

Applying Euler's equations of motions:

1st eqn: $(F - W\sin\theta)\,\underline{\hat{e}}_1 + (N_1 + N_2 - W\cos\theta)\,\underline{\hat{e}}_2 = m\,a_1\,\underline{\hat{e}}_1$

$$\Rightarrow \quad F = W\sin\theta + ma_1 \quad (1), \quad \& \quad N_1 + N_2 = W\cos\theta \quad (2)$$

2nd eqn: $\left[(N_1 - N_2)\,\ell_1 + F\ell_2\right]\underline{\hat{e}}_3 = \underline{0}$

$$\Rightarrow \quad (N_1 - N_2)\,\ell_1 + F\ell_2 = 0 \quad (3)$$

3 eqns &
3 unknowns
(can be solved!)

Applications of Euler's axioms

There are numerous applications.

A small list that we will look into:

1) Finding dynamic reaction force systems

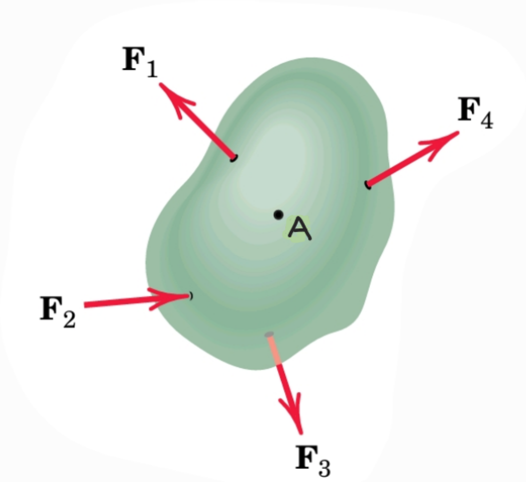
2) Rotation about a fixed RB axis

3) Plane (2D) motion of an RB

4) Balanced motion of a rotor

Steps towards solving problems on dynamics of RB

1) Identify the RB / system of interest, and represent the isolated RB/ system by FBD

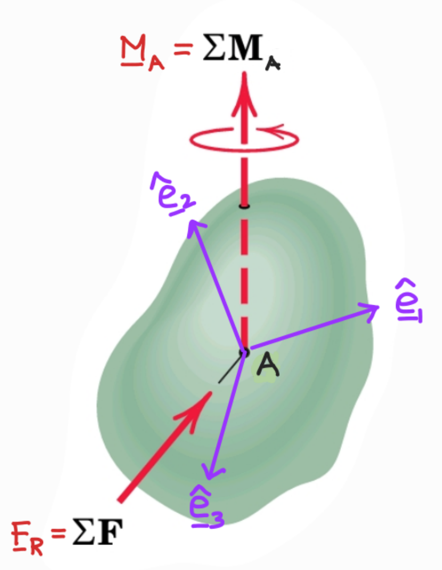


2) Identify a valid point A such that $\underline{M}_A = \frac{d}{dt}\{\underline{H}_{A|I}\}\Big|_I = \dot{\underline{H}}_{A|I}$ holds good

('I' is an inertial frame)

3) Identify a working csys $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ such that it is fixed to the RB frame 'm' and has origin at A

4> Write the equivalent force system through A



$\underline{F}_R$ = Resultant of all forces and reactions

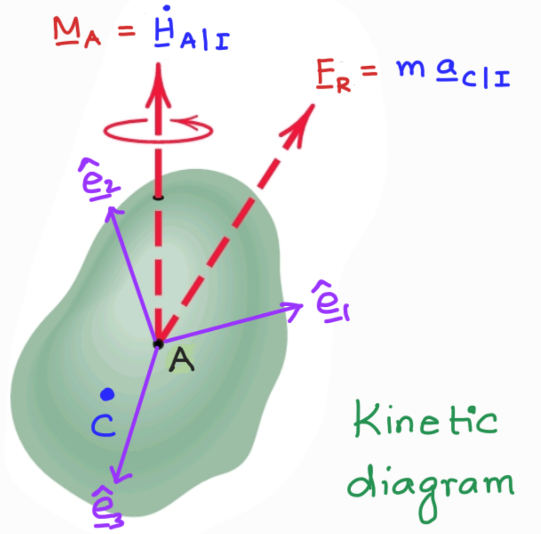
$\underline{M}_A$ = Sum of all couples and moments (due to forces) at A

5> **Choice of equations:** Identify which set of Euler's eqns is applicable I or II or Both

a> Only translation $\leftrightarrow$ I only

b> Only rotation $\leftrightarrow$ II only

c> Translation & rotation $\leftrightarrow$ Both



6> Compute the inertia matrix $[\underline{\underline{I}}^A]_{[\hat{e}_1 \hat{e}_2 \hat{e}_3]}$ using transformation & // axes theorem (no need to find components that you don't require)

7> **Kinematics** : Use kinematic relations (such as composition of angular velocities and accelerations, vel/acc transfer, rolling without slip constraints) to find expressions of $\underline{\omega}_{m|I}$, $\dot{\underline{\omega}}_{m|I}$, and $\underline{a}_{c|I}$.

e.g. if COM C is not on the axis of rotation, then use

$$\underline{a}_{c|I} = \underline{a}_{A|I} + \dot{\underline{\omega}}_{m|I} \times \underline{r}_{cA} + \underline{\omega}_{m|I} \times (\underline{\omega}_{m|I} \times \underline{r}_{cA})$$

8) Number of unknowns: In order for an RB dynamics prob. to be solvable, the number of unknowns cannot exceed the number of equations available

$\underline{\omega}_{m|I}$, $\underline{\dot{\omega}}_{m|I}$, $\underline{a}_{c|I}$

- If the motion of an RB is fully known, the Euler's equations can be used to evaluate the net forces and net moments, in particular, the unknown reactions (since they are not known in advance)

- It is possible that the number of unknown reactions to exceed the number of available independent equations. Such cases are called indeterminate systems

9) Consistency of Assumptions: While formulating the solution, the directions of certain forces/couples or accelerations may not be known in the beginning, so make an initial assumption. However, all assumptions must be made consistent with Newton's 3rd law and with any kinematic constraints.

Note: For a composite RB, $\underline{a}_{c|I}$ (for use in Euler's 1st eq)

can be found in two ways:

1st way: First find $\underline{r}_c = \dfrac{\sum_i \underline{r}_{c_i} m_i}{\sum m_i}$

COM of composite RB

COM of ith simple RB

mass of ith simple RB

2nd way: $m\,\underline{a}_{c|I} = \sum_i m_i\,\underline{a}_{c_i|I}$ ← this way is preferred where some C_i's are not accelerating

⇒ less algebra & effort

Application 1 : Finding dynamic bearing reaction forces

Given :

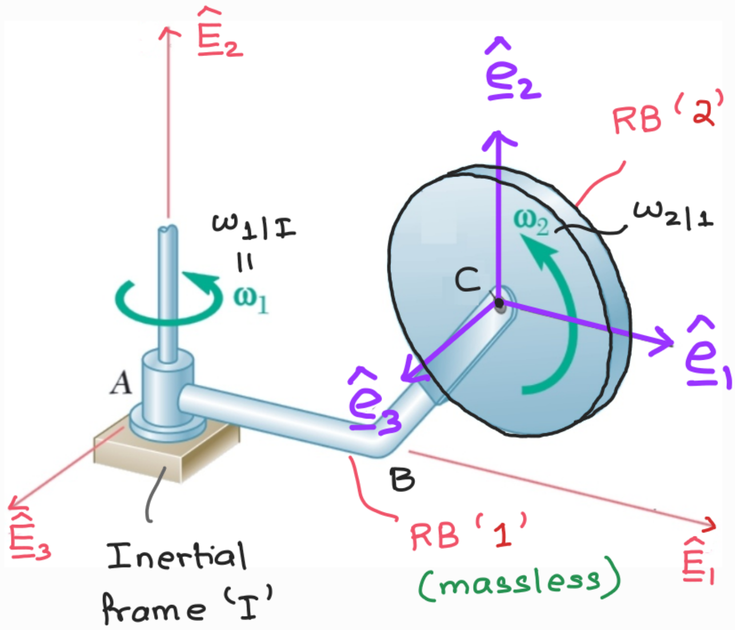
i) Link ABC is massless

ii) Disk has mass $m = 4$ kg

iii) ω_1 , ω_2 are constant angular velocities

$\omega_1 = 5$ rad/s , $\omega_2 = 15$ rad/s

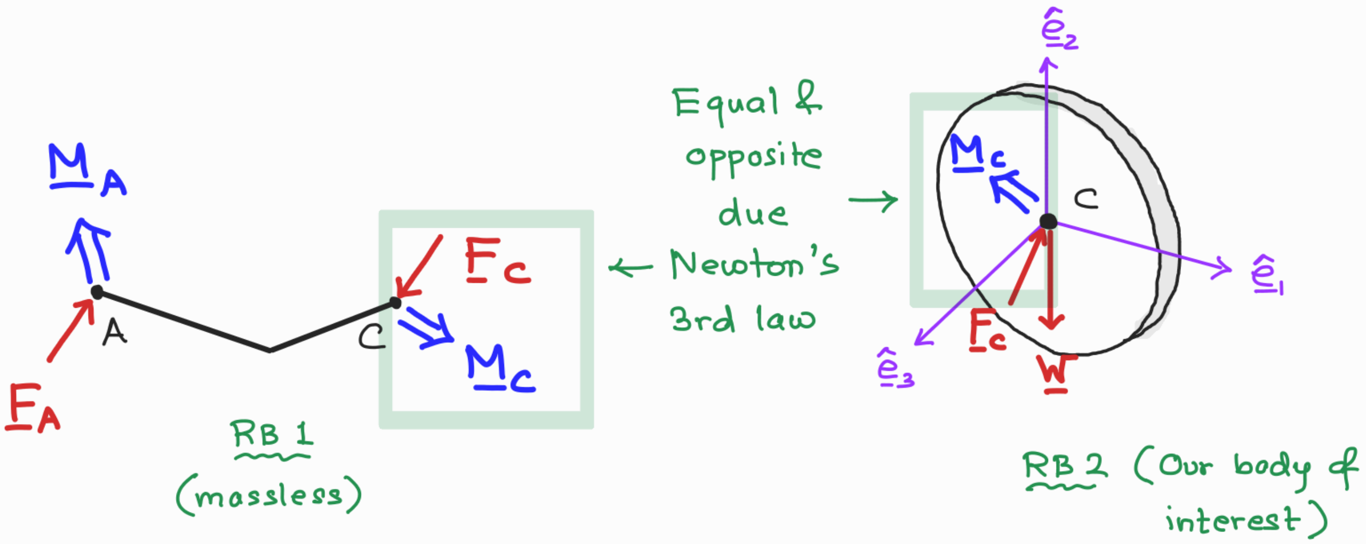
Find reaction force system exerted at support A.



Solution :

1) Identify and isolate RB of interest and draw FBD

Two simple RBs



$$\underline{F}_R = R_1 \hat{e}_1 + R_2 \hat{e}_2 + R_3 \hat{e}_3$$

$$\underline{M}_C = M_{c,1} \hat{e}_1 + M_{c,2} \hat{e}_2 + M_{c,3} \hat{e}_3$$

Since we are asked to find reaction force system at point A, which is a part of RB '1', it makes sense to start with analyzing RB '1'

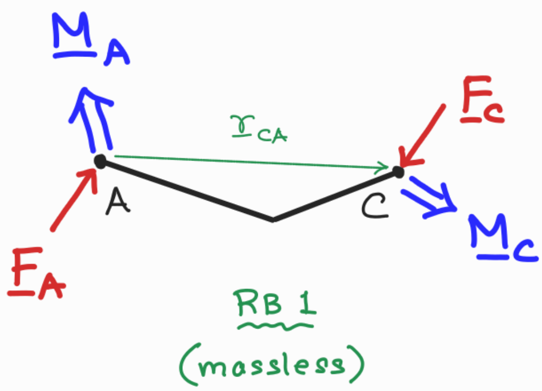
$$\underline{F}_A = F_{A,1}\,\hat{\underline{e}}_1 + F_{A,2}\,\hat{\underline{e}}_2 + F_{A,3}\,\hat{\underline{e}}_3$$

$$\underline{F}_C = F_{C,1}\,\hat{\underline{e}}_1 + F_{C,2}\,\hat{\underline{e}}_2 + F_{C,3}\,\hat{\underline{e}}_3$$

$$\underline{M}_A = M_{A,1}\,\hat{\underline{e}}_1 + M_{A,2}\,\hat{\underline{e}}_2 + M_{A,3}\,\hat{\underline{e}}_3$$

$$\underline{M}_C = M_{C,1}\,\hat{\underline{e}}_1 + M_{C,2}\,\hat{\underline{e}}_2 + M_{C,3}\,\hat{\underline{e}}_3$$

$$\underline{r}_{CA} = r_1\,\hat{\underline{e}}_1 + r_2\,\hat{\underline{e}}_2 + r_3\,\hat{\underline{e}}_3$$



Since RB '1' is massless, Euler's two equations degenerate to:

$$\underline{F}_A + \underline{F}_C = \underline{0} \qquad \text{(1st Euler eqn)}$$

$$\Rightarrow \quad \begin{aligned} F_{A,1} &= -F_{C,1} \\ F_{A,2} &= -F_{C,2} \\ F_{A,3} &= -F_{C,3} \end{aligned}$$

6 unknowns

$$\underline{M}_A + \underline{M}_C + \underline{r}_{CA} \times \underline{F}_C = \underline{0} \qquad \text{(2nd Euler eqn)}$$

+

$$\Rightarrow \quad \begin{aligned} M_{A,1} + M_{C,1} + (r_2 F_{C,3} - r_3 F_{C,2}) &= 0 \\ M_{A,2} + M_{C,2} + (r_3 F_{C,1} - r_1 F_{C,3}) &= 0 \\ M_{A,3} + M_{C,3} + (r_1 F_{C,2} - r_2 F_{C,1}) &= 0 \end{aligned}$$

(*)

6 unknowns

6 equations

We have 12 unknowns (all components of $\underline{M}_A$, $\underline{M}_C$, $\underline{F}_A$, $\underline{F}_C$) and only 6 equations. We need more equations.

Lets move to RB '2'

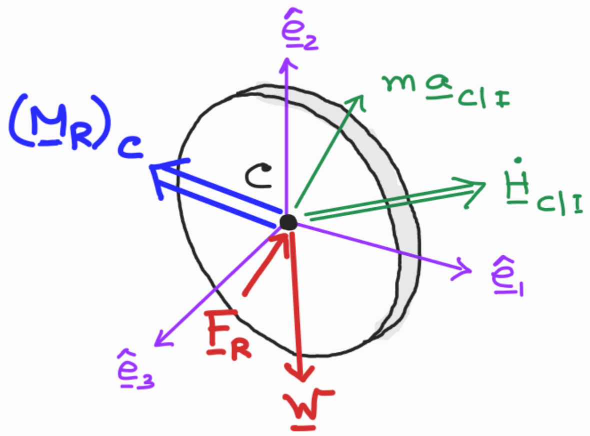
- Point C which is the COM of the disc is a valid point for $\dot{\underline{H}}_{C|I} = \underline{M}_C$

- Fix our working csys $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ on the disc with C as origin. Note $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ coincide with the principal axes of the disk.

- Find equivalent force system at C

$$(\underline{F}_R)_C = R_1\,\hat{\underline{e}}_1 + (R_2 - mg)\,\hat{\underline{e}}_2 + R_3\,\hat{\underline{e}}_3$$

$$\underline{M}_C = M_{C,1}\,\hat{\underline{e}}_1 + M_{C,2}\,\hat{\underline{e}}_2 + M_{C,3}\,\hat{\underline{e}}_3$$



Kinetic diagram

- Since we need to find both reaction forces and couples, we require both of Euler's equations.

For forces $\longrightarrow$ use Euler's 1st set

$$\underline{F}_c = m\, \underline{a}_{c|I}$$

$$\Rightarrow F_{c,1}\, \hat{e}_1 + (F_{c,2} - mg)\, \hat{e}_2 + F_{c,3}\, \hat{e}_3 = m \left[a_{c,1}\, \hat{e}_1 + a_{c,2}\, \hat{e}_2 + a_{c,3}\, \hat{e}_3 \right]$$

$$\Rightarrow \begin{cases} F_{c,1} = m\, a_{c,1} \\ F_{c,2} = m\, a_{c,2} + mg \\ F_{c,3} = m\, a_{c,3} \end{cases} \quad (**)$$

3 more unknowns $(\underline{a}_{c|I})$

For couples, use Euler's 2nd and set

$$\underline{M}_c = M_{c,1}\, \hat{e}_1 + M_{c,2}\, \hat{e}_2 + M_{c,3}\, \hat{e}_3 = \underline{\underline{I}}^c\, \dot{\underline{\omega}}_{2|I}$$

Can we use any simplified forms of Euler's 2nd equation?

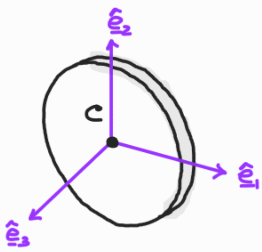
YES!

Simplified cases of Euler's 2nd axiom

① When $\hat{e}_1$-$\hat{e}_2$-$\hat{e}_3$ are principal axes at A:

$$\Rightarrow [\underline{\underline{I}}^A]_{(\hat{e}_1-\hat{e}_2-\hat{e}_3)} = \begin{bmatrix} I_{11}^A & 0 & 0 \\ 0 & I_{22}^A & 0 \\ 0 & 0 & I_{33}^A \end{bmatrix}$$

$$\begin{aligned} \underline{M}^A = & \left[I_{11}^A\, \dot{\omega}_1 - (I_{22}^A - I_{33}^A)\, \omega_2 \omega_3 \right] \hat{e}_1 \\ & + \left[I_{22}^A\, \dot{\omega}_2 - (I_{33}^A - I_{11}^A)\, \omega_3 \omega_1 \right] \hat{e}_2 \\ & + \left[I_{33}^A\, \dot{\omega}_3 - (I_{11}^A - I_{22}^A)\, \omega_1 \omega_2 \right] \hat{e}_3 \end{aligned}$$



i) $\underline{\omega}_{2|I} = \underline{\omega}_{1|I} + \underline{\omega}_{2|1} = \omega_1\, \hat{e}_2 + \omega_2\, \hat{e}_3$

ii) $\dot{\underline{\omega}}_{2|I} = \overset{0}{\dot{\underline{\omega}}_{1|I}} + \overset{0}{\dot{\underline{\omega}}_{2|1}} + \underline{\omega}_{1|I} \times \underline{\omega}_{2|1} = \omega_1 \omega_2\, \hat{e}_1$

constant (given)

So the Euler's 2nd set of equations are:

$$M_{c,1} = I_{11}^{c} \cancelto{\omega_1\omega_2}{(\dot{\omega}_{2|I})_1} - (I_{22}^{c} - I_{33}^{c}) \cancelto{\omega_1}{(\omega_{2|I})_2} \cancelto{\omega_2}{(\omega_{2|I})_3}$$

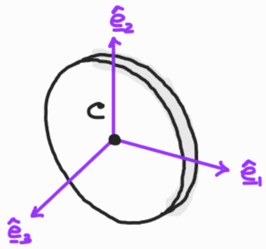
$$M_{c,2} = I_{22}^{c} \cancelto{0}{(\dot{\omega}_{2|I})_2} - (I_{33}^{c} - I_{11}^{c}) \cancelto{\omega_2}{(\omega_{2|I})_3} \cancelto{0}{(\omega_{2|I})_1}$$

$$M_{c,3} = I_{33}^{c} \cancelto{0}{(\dot{\omega}_{2|I})_3} - (I_{11}^{c} - I_{22}^{c}) \cancelto{0}{(\omega_{2|I})_1} \cancelto{\omega_1}{(\omega_{2|I})_2}$$

Need to determine all three I_{11}^c, I_{22}^c, and I_{33}^c

The thin disc being body of revolution abt $\hat{e}_3$ $\Rightarrow$ $I_{11}^c = I_{22}^c$

$$= \frac{m r^2}{4} = \frac{(4)(0.15)^2}{4} = 0.225 \text{ kg m}^2$$



and $I_{33}^c = I_{11}^c + I_{22}^c$ (Recall ⊥ axes thm for thin lamina)

$$= \frac{m r^2}{2} = 0.45 \text{ kg m}^2$$

Therefore,

$$\begin{cases} M_{c,1} = I_{11}^{c}\, \omega_1\omega_2 - (I_{22}^{c} - I_{33}^{c})\, \omega_1\omega_2 \\ \quad\;\;\; = (\cancel{I_{11}^{c}} - \cancel{I_{22}^{c}} + I_{33}^{c})\, \omega_1\omega_2 = (0.45)(5)(15) = 33.75 \text{ Nm} \\ M_{c,2} = 0 \\ M_{c,3} = 0 \end{cases}$$

$\underline{M}_c$ known ✓

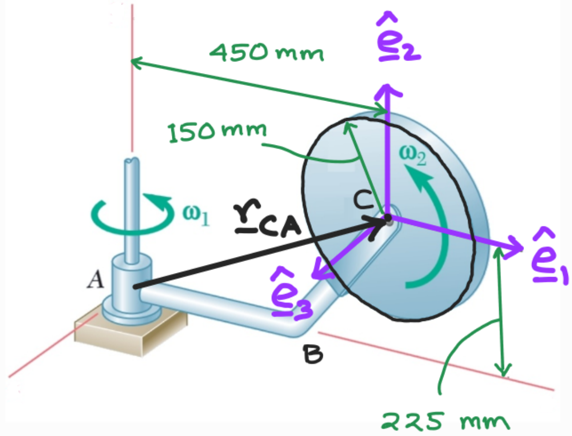
$\underline{M}_A$, $\underline{F}_A$, $\underline{F}_c$, $\underline{a}_{c|I}$?

(12 unknowns)

We are now left to use kinematics relations

Linear acc. of COM

$$\begin{cases} \underline{a}_{C|I} = \cancel{\underline{a}_{A|I}}^{0} + \cancel{\dot{\underline{\omega}}_{1|I}}^{0} \times \underline{r}_{CA} \\ \qquad + \underline{\omega}_{1|I} \times (\underline{\omega}_{1|I} \times \underline{r}_{CA}) \\ \underline{r}_{CA} = 0.45\,\hat{\underline{e}}_1 + 0.225\,\hat{\underline{e}}_2 \end{cases}$$



$$\therefore \quad \underline{a}_{C|I} = (\omega_1\,\hat{\underline{e}}_2) \times \left\{ (\omega_1\,\hat{\underline{e}}_2) \times (0.45\,\hat{\underline{e}}_1 + 0.225\,\hat{\underline{e}}_2) \right\}$$

$$= -0.45\,\omega_1^2\,\hat{\underline{e}}_1$$

Plugging in the values of $\omega_1 = 5\ \text{rad/s}$ and $\omega_2 = 15\ \text{rad/s}$

$$\underline{a}_{C|I} = -11.25\,\hat{\underline{e}}_1 \ (m/s^2)$$

$\underline{a}_{C|I}$ known ✓

Plug the above value in (**) and get $\underline{F}_C$

$$F_{C,1} = m\,a_{C,1} = -45\ \text{N}$$

$$F_{C,2} = m\,\cancel{a_{C,2}}^{0} + mg = 4\,(9.81) = 39.24\ \text{N}$$

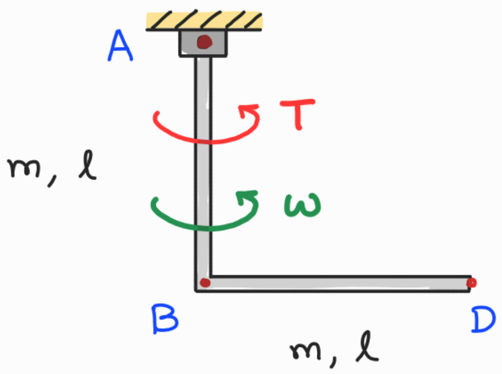
$$F_{C,3} = m\,\cancel{a_{C,3}}^{0} = 0$$

$\underline{F}_C$ known ✓

You now have $\underline{M}_A$, $\underline{F}_A$ (6 unknowns) & 6 eqns

(*), solve and get them!

Application 2: Rotation about a fixed RB axis



Given:

ⓐ L-shaped composite RB: two thin simple rods of length ℓ and mass m

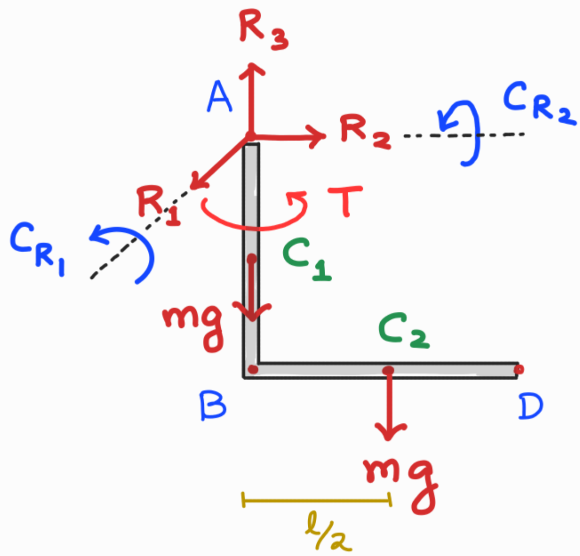
ⓑ A torque T is applied on it

Find (i) angular velocity $\omega(t)$

(ii) reaction forces and moments at A

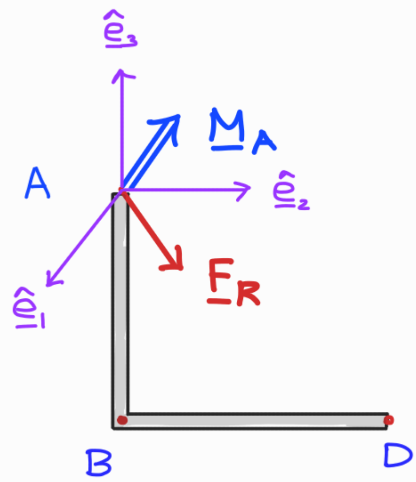
Solution

- Draw FBD by isolating the composite RB from supports



- Point A is fixed to the inertial ground frame, so pt A is a valid point!

- Fix our working csys $\hat{e}_1$-$\hat{e}_2$-$\hat{e}_3$ on the RB



- Find equivalent force system at A

$$\underline{F}_R = R_1\,\hat{\underline{e}}_1 + R_2\,\hat{\underline{e}}_2 + (-2mg + R_3)\,\hat{\underline{e}}_3$$

$$\underline{M}_A = \left(C_{R_1} - mg\frac{\ell}{2}\right)\hat{\underline{e}}_1 + C_{R_2}\,\hat{\underline{e}}_2 + T\,\hat{\underline{e}}_3 \leftarrow$$ (No couples due to R_1, R_2, R_3 as they pass through A)

– Unknowns are : $\underline{\omega}_{m|I} = \omega\,\hat{\underline{e}}_3$ (other components are given as zero)

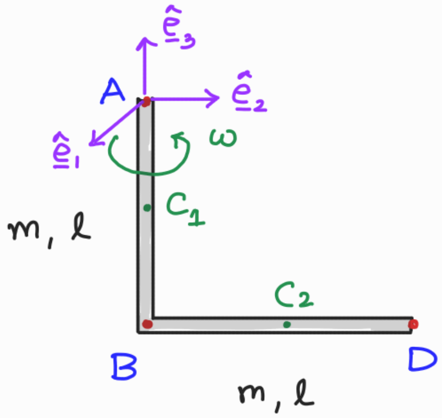
$\dot{\underline{\omega}}_{m|I} = \dot{\omega}\,\hat{\underline{e}}_3$ (since $\omega_1 = \omega_2 = 0$)

$\underline{a}_{c|I}$ (all components)

– Choice of Euler's equations: Both

Using Euler's 1st equation (for translational motion)

$$\underline{F}_R = \sum_{i=1}^{2} m_i\,\underline{a}_{c_i|I} = m_1 \cancelto{0}{\underline{a}_{c_1|I}} + \cancelto{m}{m_2}\,\underbrace{\underline{a}_{c_2|I}}_{?}$$



$\Downarrow$

To obtain $\underline{a}_{C_2|I}$, you can use acc. transfer relations from kinematics

I.P. ≡ B , I.F ≡ 'm' ← (RB)

$$\underline{a}_{C_2|I} = \cancelto{0}{\underline{a}_{B|I}} + \cancelto{\dot{\omega}\hat{\underline{e}}_3}{\dot{\underline{\omega}}_{m|I}} \times \cancelto{\frac{\ell}{2}\hat{\underline{e}}_2}{\underline{r}_{C_2B}}$$

$$+ \cancelto{\omega\hat{\underline{e}}_3}{\underline{\omega}_{m|I}} \times \left(\cancelto{\omega\hat{\underline{e}}_3}{\underline{\omega}_{m|I}} \times \cancelto{\frac{\ell}{2}\hat{\underline{e}}_2}{\underline{r}_{C_2B}}\right)$$

$$= -\frac{\ell}{2}\omega^2\,\hat{\underline{e}}_2$$

$$\therefore \quad \underline{E}_R = -m\frac{l}{2}\dot{\omega}\,\hat{\underline{e}}_1 - m\frac{l}{2}\omega^2\,\hat{\underline{e}}_2$$

$$\Rightarrow \quad R_1\,\hat{\underline{e}}_1 + R_2\,\hat{\underline{e}}_2 + (-2mg + R_3)\,\hat{\underline{e}}_3 = -m\frac{l}{2}\dot{\omega}\,\hat{\underline{e}}_1 - m\frac{l}{2}\omega^2\,\hat{\underline{e}}_2$$

Upon comparing the components :

1a — $R_1 = -m\frac{l}{2}\dot{\omega}$

1b — $R_2 = -m\frac{l}{2}\omega^2$

still unknown

1c — $R_3 = 2mg$ ✓

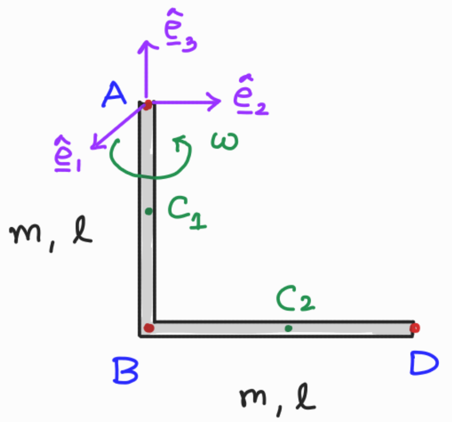
Let's use Euler's 2nd equation for rotational motion.

$$\underline{M}_A = \underline{\underline{I}}^A\,\dot{\underline{\omega}}_{m|I} - \underline{\omega}_{m|I} \times (\underline{\underline{I}}^A\,\underline{\omega}_{m|I})$$

Can we use any simplified forms of Euler's 2nd equation?

YES!

Recall Case 2 from Lec 13



② **Rotation of RB about an RB-fixed axis** $(\underline{\omega}_{m|I} = \omega\,\hat{\underline{e}}_3)$ (say $\hat{e}_3$)

Here, the RB is constrained to rotate about an RB-fixed axis, say $\hat{\underline{e}}_3$, so that $\underline{\omega}_{m|I} = \omega\,\hat{\underline{e}}_3$ and $\dot{\underline{\omega}}_{m|I} = \dot{\omega}\,\hat{\underline{e}}_3$

The axes $\hat{e}_1$-$\hat{e}_2$-$\hat{e}_3$ may not necessarily be principal axes

Here, the RB rotates about an RB-fixed axis $\hat{\underline{e}}_3$

$\Downarrow$

follows Case 2

For Case 2, the Euler's 2nd equation was simplified as:

$$M_{A,1} = I_{13}^{A}\,\dot{\omega} - I_{23}^{A}\,\omega^2$$

$$M_{A,2} = I_{23}^{A}\,\dot{\omega} + I_{13}^{A}\,\omega^2$$

$$M_{A,3} = I_{33}^{A}\,\dot{\omega}$$

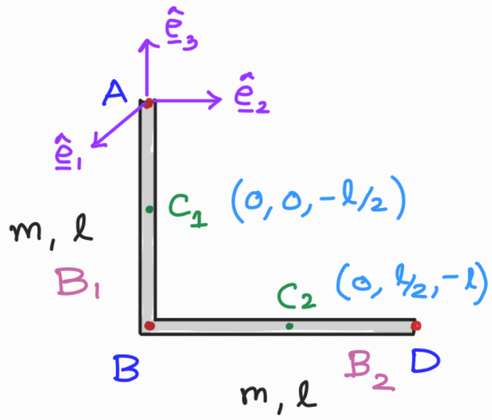
Compute these next!

Calculate the required inertia tensor components treating A as the origin and RB-fixed csys $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$

$B_1 \equiv$ rod AB

$B_2 \equiv$ rod BD

- $\boxed{I_{13}^{A}}$ for composite RB (recall Lec 12)



$$I_{13}^{A} = I_{13}^{A}(B_1) + I_{13}^{A}(B_2)$$

$$I_{13}^{A}(B_1) = \cancelto{0}{I_{13}^{C_1}(B_1)} - m\,\cancelto{0}{(x_{c_1})_1}\,(x_{c_1})_3 = 0$$

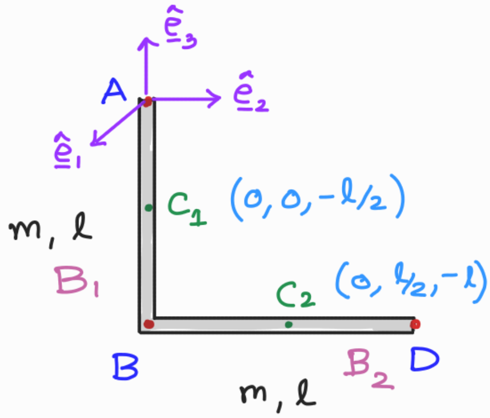
$(x_1 \approx 0)$

$$I_{13}^{A}(B_2) = \cancelto{0}{I_{13}^{C_2}(B_2)} - m\,\cancelto{0}{(x_{c_2})_1}\,(x_{c_2})_3 = 0$$

$\begin{pmatrix} x_1 \approx 0 \\ x_3 \approx 0 \end{pmatrix}$

$\therefore$ I_{13}^{A} (for composite 'L' shaped RB) $= 0$

- $\boxed{I_{23}^{A}}$ for composite RB (recall Lec 12)



$$I_{23}^{A} = I_{23}^{A}(B_1) + I_{23}^{A}(B_2)$$

$$I_{23}^{A}(B_1) = \underbrace{\cancelto{0}{I_{23}^{C_1}(B_1)}}_{(x_2 \approx 0)} - m \cancelto{0}{(x_{C_1})_2} (x_{C_1})_3 = 0$$

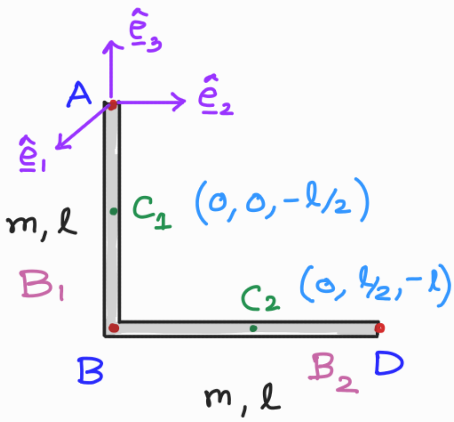
$$I_{23}^{A}(B_2) = \underbrace{\cancelto{0}{I_{23}^{C_2}(B_2)}}_{(x_3 \approx 0)} - m \cancelto{\ell/2}{(x_{C_2})_2} \cancelto{-\ell}{(x_{C_2})_3} = \frac{m\ell^2}{2}$$

position of C_2 from A in $\hat{e}_3$ dir

position " " " " " $\hat{e}_2$ dir

$$\therefore \quad I_{23}^{A} \text{ (for composite 'L' shaped RB)} = \frac{m\ell^2}{2}$$

- $\boxed{I_{33}^{A}}$ for composite RB



$$I_{33}^{A} = I_{33}^{A}(B_1) + I_{33}^{A}(B_2)$$

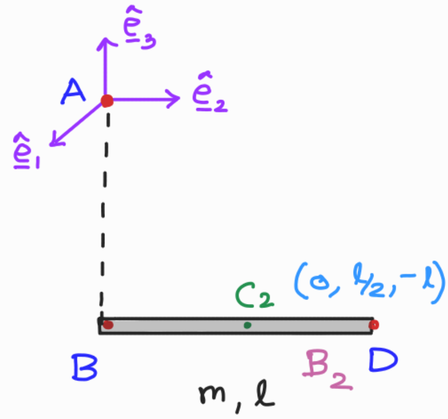
$$I_{33}^{A}(B_1) = I_{33}^{C_1}(B_1) + m\left\{(x_{C_1})_1^2 + (x_{C_1})_2^2\right\}$$

$$= 0 + m\left\{0^2 + 0^2\right\} = 0$$

$$I_{33}^{A}(B_2) = I_{33}^{C_2}(B_1) + m\left\{(x_{C_2})_1^2 + (x_{C_2})_2^2\right\}$$

$$= \frac{m\ell^2}{12} + m\left\{0^2 + \left(-\frac{\ell}{2}\right)^2\right\}$$

$$= \frac{m\ell^2}{12} + \frac{m\ell^2}{4} = \frac{m\ell^2}{3}$$



$\therefore$ I^A_{33} (for composite 'L' shaped RB) $= \frac{m\ell^2}{3}$

Plugging the inertia matrix components in the simplified Euler's equations lead to :

$$M_{A,1} = I^A_{13}\,\dot{\omega} - I^A_{23}\,\omega^2 = -\frac{m\ell^2}{2}\omega^2$$

$$M_{A,2} = I^A_{23}\,\dot{\omega} + I^A_{13}\,\omega^2 = \frac{m\ell^2}{2}\dot{\omega}$$

$$M_{A,3} = I^A_{33}\,\dot{\omega} = \frac{m\ell^2}{3}\dot{\omega}$$

Recall : $\underline{M}_A = \underbrace{\left(C_{R_1} - mg\frac{\ell}{2}\right)}_{M_{A,1}} \hat{\underline{e}}_1 + \underbrace{C_{R_2}}_{M_{A,2}} \hat{\underline{e}}_2 + \underbrace{T}_{M_{A,3}} \hat{\underline{e}}_3$

Comparing the components, we get :

$$C_{R_1} - mg\,\ell/2 = -\frac{m\ell^2}{2}\omega^2 \quad - 2a$$

$$C_{R_2} = \frac{m\ell^2}{2}\dot{\omega} \quad - 2b$$

$$T = \frac{m\ell^2}{3}\dot{\omega} \quad - 2c$$

Solving for SIX unknowns : $\omega / \dot{\omega}$, C_{R_1} , C_{R_2}, R_1, R_2, R_3

(2c) $\Rightarrow$ $$\boxed{\dot{\omega} = \frac{3T}{ml^2}}$$ $\rightarrow$ $$\frac{d\omega}{dt} = \frac{3T}{ml^2}$$

$$\Rightarrow \int_{\omega(0) = \omega_0}^{\omega(t)} d\omega(t) = \frac{3T}{ml^2} \int_{t=0}^{t=t} dt$$

$$\Rightarrow \omega(t) - \omega_0 = \frac{3T}{ml^2} t$$

$$\Rightarrow \omega(t) = \frac{3T}{ml^2} t + \underline{\underline{\omega_0}}$$

Initial condition

$$\boxed{\omega(t) = \frac{3T}{ml^2} t + \omega_0}$$

(2b) $$\Rightarrow C_{R_2} = \frac{ml^2}{2} \dot{\omega} = \frac{ml^2}{2} \left(\frac{3T}{ml^2} \right) = \frac{3T}{2}$$

(2c) $$\Rightarrow C_{R_1} = -\frac{ml^2}{2} \omega^2 + \frac{mgl}{2} = \frac{mgl}{2} - \frac{ml^2}{2} \left(\frac{3T}{ml^2} t + \omega_0 \right)^2$$

(1a) $$\Rightarrow R_1 = -\frac{ml}{2} \dot{\omega} = -\frac{\cancel{ml}}{2} \left(\frac{3T}{\cancel{ml^2}} \right) = \frac{3T}{2l}$$

(1b) $$\Rightarrow R_2 = -\frac{ml}{2} \omega^2 = -\frac{ml}{2} \left(\frac{3T}{ml^2} t + \omega_0 \right)^2$$

(1c) $$\Rightarrow R_3 = 2mg$$

In this problem, Six unknowns could be determined by using the two sets of Euler's equations (Six scalar eqns), but there are several cases where the # of unknowns exceed the six equations.