Recap

In the last 2-3 lectures, we have been talking a lot about Euler's 2nd axiom

- How it was defined for a point O fixed in the inertial frame 'I'

$$\dot{\underline{H}}_{O|I} = \underline{M}_O \qquad (O \in I) \quad \text{— } ①$$

- Then we wanted to derive a modified Euler's 2nd axiom about a point A moving (not fixed) in 'I' and we arrived at the relation

$$\frac{d}{dt}\{\underline{H}_{A|I}\}\Big|_I = \dot{\underline{H}}_{A|I} = \underline{M}_A - \underline{r}_{CA} \times m\, \underline{a}_{A|I} \quad \text{— } ②$$

where C is the COM of the RB

The objective then was to identify ALL such points A for which Euler's 2nd axiom retains the same basic form as ①

$$\dot{\underline{H}}_{A|I} = \underline{M}_A \quad \text{if and only if}$$

1) $A \equiv C \rightarrow$ COM $\quad (\underline{r}_{CA} = \underline{0})$ → Most general case!

and/or

2) $\underline{a}_{A|I} = 0 \rightarrow$ point A is fixed in 'I'

and/or

3) $\underline{a}_{A|I} \;//\; \underline{r}_{CA} \rightarrow$ acc. of A is directed through COM of RB

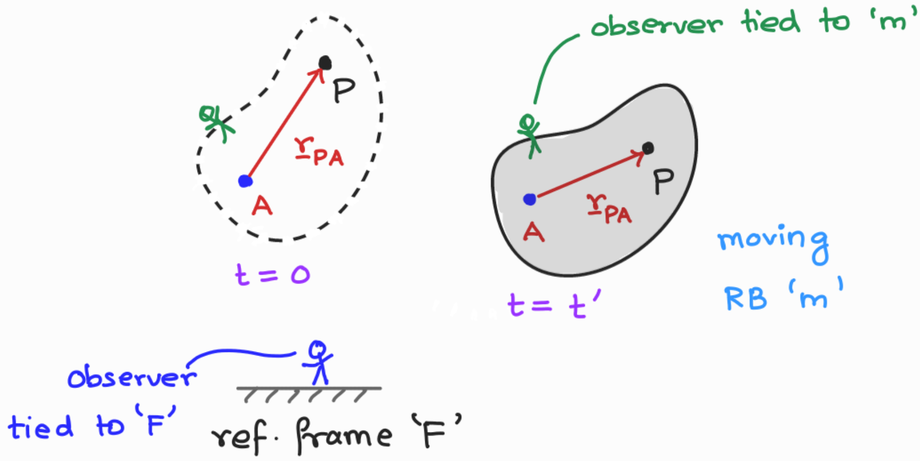
Variation of Inertia Tensor with time

$$\underline{\underline{I}}^A = \int_m \left\{ (\underline{r}_{PA} \cdot \underline{r}_{PA}) \underline{\underline{I}} - \underline{r}_{PA} \otimes \underline{r}_{PA} \right\} dm$$

Observe that only the reference point A needs to be fixed on the RB (or its massless extension) for the calculation of $\underline{\underline{I}}^A$, but the reference frame 'F' need not be fixed to the RB

a) Fixing pt A on RB $\Rightarrow$ $|\underline{r}_{PA}|$ (the magnitude) is constant

b) But the orientation of $\underline{r}_{PA}$ can still change as the RB rotates relative to frame 'F'

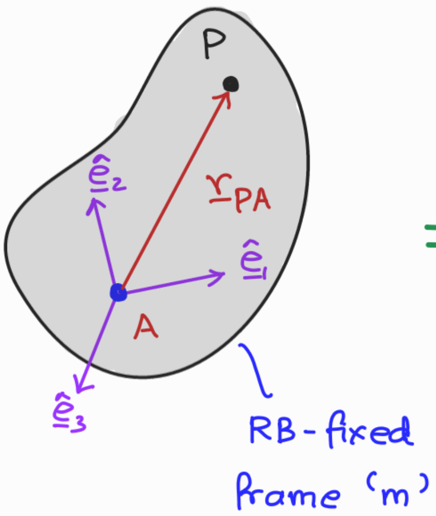


orientation of $\underline{r}_{PA}$ changes with time relative to an observer in 'F'

$\Rightarrow$ Inertia Tensor will vary with time relative to frame 'F'

But for a moving ref. frame 'm', which is a frame attached to the RB itself, the inertia tensor $\underline{\underline{I}}^A$ is a CONSTANT tensor.

The matrix components of $\underline{\underline{I}}^A$ will depend only on the choice of orientation of csys associated with the RB-fixed frame 'm'



$\Rightarrow$

$\underline{\underline{I}}^A \rightarrow$ Constant tensor relative to RB-fixed frame 'm'

$$\therefore \quad \frac{d}{dt}\left\{\underline{\underline{I}}^A\right\}\Big|_m = \underline{\underline{0}} \quad \text{(zero tensor)}$$

$\left[\underline{\underline{I}}^A\right]_{\left[\begin{smallmatrix}\hat{\underline{e}}_1 \\ \hat{\underline{e}}_2 \\ \hat{\underline{e}}_3\end{smallmatrix}\right]}$ components will depend only on orientation of $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$ in frame 'm'

Euler's 2nd axiom in terms of Inertia Tensor

Having derived the following :

1) Angular momentum abt pt A as a product of inertia tensor at A and angular velocity of RB

$$\underline{H}_{A|F} = \underline{\underline{I}}^A \, \underline{\omega}_{m|F}$$

replace F with inertial frame 'I'

$$\underline{H}_{A|I} = \underline{\underline{I}}^A \, \underline{\omega}_{m|I} \quad \text{—— } (1)$$

2) Modified Euler's 2nd axiom

$$\frac{d}{dt}\left\{\underline{H}_{A|I}\right\}\Big|_I = \dot{\underline{H}}_{A|I} = \underline{M}_A - \underline{r}_{CA} \times m\,\underline{a}_{A|I}$$

$C \equiv$ COM of RB

our objective is to obtain modified Euler's 2nd axiom (also known as Angular Momentum Balance equation) for an RB using the inertia tensor $\underline{\underline{I}}^A$.

Let's consider a point 'A' on the RB where

$$(2) \text{ —— } \dot{\underline{H}}_{A|I} = \underline{M}_A$$

$$\Rightarrow \frac{d}{dt}\left\{\underline{H}_{A|I}\right\}\Big|_I = \underline{M}_A$$

if $\begin{cases} A \equiv C \rightarrow \text{COM} \\ \underline{a}_{A|I} = \underline{0} \\ \underline{a}_{A|I} \text{ is directed through COM} \end{cases}$

Recall that the time derivative of a vector in two different frames — fixed frame 'F' and moving frame 'm' — was related:

$$\left.\frac{d\underline{A}}{dt}\right|_F = \left.\frac{d\underline{A}}{dt}\right|_m + \underline{\omega}_{m|F} \times \underline{A}$$

'F' and 'm' can also be flipped around:

$$\left.\frac{d\underline{A}}{dt}\right|_m = \left.\frac{d\underline{A}}{dt}\right|_F + \underline{\omega}_{F|m} \times \underline{A}$$

Let's write the time derivative of $\underline{H}_{A|I}$ relative to 'I' using equation ①

Treat $\underline{A} \equiv \underline{H}_{A|I}$

$$\left.\frac{d}{dt}\{\underline{H}_{A|I}\}\right|_I = \left.\frac{d}{dt}\{\underline{H}_{A|I}\}\right|_m + \underbrace{\underline{\omega}_{m|I} \times \underline{H}_{A|I}}_{\underline{\omega}_{m|I} \times (\underline{\underline{I}}^A \underline{\omega}_{m|I})}$$

$$\left.\frac{d}{dt}\{\underline{\underline{I}}^A \underline{\omega}_{m|I}\}\right|_m = \cancel{\left.\frac{d}{dt}\{\underline{\underline{I}}^A\}\right|_m}^{\,\underline{0}} \underline{\omega}_{m|I} + \underline{\underline{I}}^A \left.\frac{d}{dt}\{\underline{\omega}_{m|I}\}\right|_m$$

$$= \underline{0} + \underline{\underline{I}}^A \left\{ \left.\frac{d}{dt}\{\underline{\omega}_{m|I}\}\right|_I + \underbrace{\underline{\omega}_{I|m}}_{= -\underline{\omega}_{m|I}} \times \underline{\omega}_{m|I} \right\}$$

$$= \underline{0} + \underline{\underline{I}}^A \dot{\underline{\omega}}_{m|I} - \cancel{\underline{\omega}_{m|I} \times \underline{\omega}_{m|I}}^{\,\underline{0}}$$

$$= \underline{\underline{I}}^A \dot{\underline{\omega}}_{m|I}$$

Thus,

$$\dot{\underline{H}}_{A|I} = \underline{\underline{I}}^A \, \dot{\underline{\omega}}_{m|I} + \underline{\omega}_{m/I} \times \left(\underline{\underline{I}}^A \, \underline{\omega}_{m|I}\right)$$

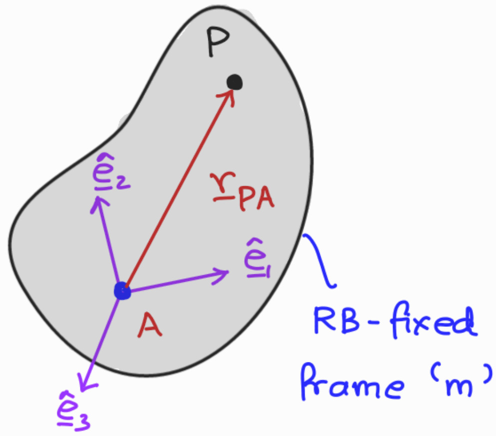
Note: The inertia tensor stays the same in RB-fixed frame 'm'

Using equation (2), we can write:

$$\boxed{\underline{\underline{I}}^A \, \dot{\underline{\omega}}_{m|I} + \underline{\omega}_{m|I} \times \left(\underline{\underline{I}}^A \, \underline{\omega}_{m|I}\right) = \underline{M}_A}$$

Expressing $\underline{\omega}_{m|I}$ and $\dot{\underline{\omega}}_{m|I}$ in terms of RB-fixed coordinate system $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$

$$\left[\underline{\omega}_{m|I}\right]_{\begin{pmatrix}\hat{\underline{e}}_1\\ \hat{\underline{e}}_2\\ \hat{\underline{e}}_3\end{pmatrix}} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$



or, $\underline{\omega}_{m|I} = \omega_1 \hat{\underline{e}}_1 + \omega_2 \hat{\underline{e}}_2 + \omega_3 \hat{\underline{e}}_3$

$$\left[\dot{\underline{\omega}}_{m|I}\right]_{\begin{pmatrix}\hat{\underline{e}}_1\\ \hat{\underline{e}}_2\\ \hat{\underline{e}}_3\end{pmatrix}} = \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix}$$

or, $\dot{\underline{\omega}}_{m|I} = \dot{\omega}_1 \hat{\underline{e}}_1 + \dot{\omega}_2 \hat{\underline{e}}_2 + \dot{\omega}_3 \hat{\underline{e}}_3$

we get a GENERAL set of three coupled nonlinear (in $\underline{\omega}$) ODEs:

$$\begin{bmatrix} I^A_{11} & I^A_{12} & I^A_{13} \\ I^A_{21} & I^A_{22} & I^A_{23} \\ I^A_{31} & I^A_{32} & I^A_{33} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \times \left(\begin{bmatrix} I^A_{11} & I^A_{12} & I^A_{13} \\ I^A_{21} & I^A_{22} & I^A_{23} \\ I^A_{31} & I^A_{32} & I^A_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \right) = \begin{bmatrix} M_{A,1} \\ M_{A,2} \\ M_{A,3} \end{bmatrix}$$

The above system of ODEs are Euler's equations for rotational motion of a rigid body.

Simplified cases of Euler's 2nd axiom

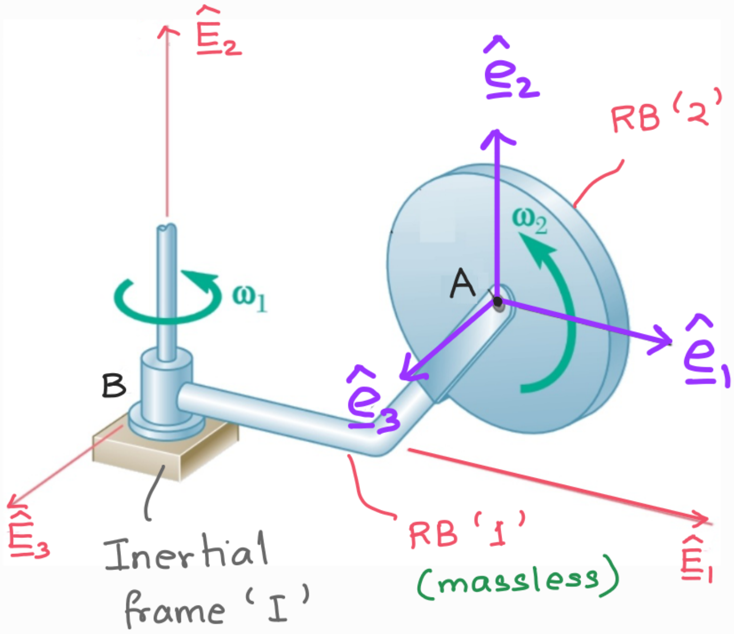
① When $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$ are principal axes at A:

$$\Rightarrow [\underline{\underline{I}}^A]_{(\hat{e}_1-\hat{e}_2-\hat{e}_3)} = \begin{bmatrix} I_{11}^A & 0 & 0 \\ 0 & I_{22}^A & 0 \\ 0 & 0 & I_{33}^A \end{bmatrix}$$

Note:

- $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$ is fixed to the disc (RB '2') in example and rotates with it
- Rod AB is a massless RB
- Point B is fixed to an inertial frame 'I'
- Point A of the disc coincides with the COM of the disc

Example:



- The disc can be thought as a body of revolution with $\hat{e}_3$-axis being the axis of revolution

Therefore, $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$ axes are also the principal axes and in this case $I_{11}^A = I_{22}^A$!

Euler's 2nd axiom (or angular momentum balance) reduces to:

$$\begin{bmatrix} I_{11}^A & 0 & 0 \\ 0 & I_{22}^A & 0 \\ 0 & 0 & I_{33}^A \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \times \left(\begin{bmatrix} I_{11}^A & 0 & 0 \\ 0 & I_{22}^A & 0 \\ 0 & 0 & I_{33}^A \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \right) = \begin{bmatrix} M_{A,1} \\ M_{A,2} \\ M_{A,3} \end{bmatrix}$$

$$\begin{bmatrix} I_{11}^A \, \dot{\omega}_1 \\ I_{22}^A \, \dot{\omega}_2 \\ I_{33}^A \, \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} -\left(I_{22}^A - I_{33}^A\right) \omega_2 \omega_3 \\ -\left(I_{33}^A - I_{11}^A\right) \omega_3 \omega_1 \\ -\left(I_{11}^A - I_{22}^A\right) \omega_1 \omega_2 \end{bmatrix} = \begin{bmatrix} M_{A,1} \\ M_{A,2} \\ M_{A,3} \end{bmatrix}$$

Therefore, the simplified Euler's 2nd axiom (when the RB-fixed csys coincides with the principal axes triad at pt A) is:

$$\begin{aligned} \underline{M}_A = & \left[I_{11}^A \, \dot{\omega}_1 - \left(I_{22}^A - I_{33}^A \right) \omega_2 \omega_3 \right] \hat{\underline{e}}_1 \\ & + \left[I_{22}^A \, \dot{\omega}_2 - \left(I_{33}^A - I_{11}^A \right) \omega_3 \omega_1 \right] \hat{\underline{e}}_2 \\ & + \left[I_{33}^A \, \dot{\omega}_3 - \left(I_{11}^A - I_{22}^A \right) \omega_1 \omega_2 \right] \hat{\underline{e}}_3 \end{aligned}$$

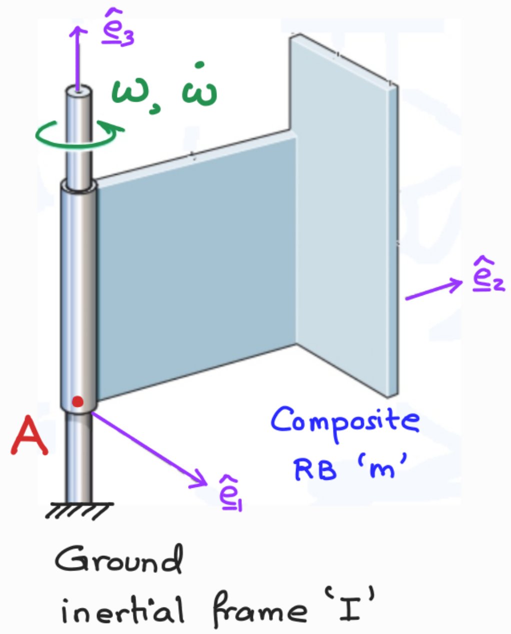
$\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ are along principal axes of $\underline{\underline{I}}^A$ at A

② Rotation of RB about an RB-fixed axis (say $\hat{e}_3$) $\left(\underline{\omega}_{m|I} = \omega\, \hat{e}_3\right)$

Here, the RB is constrained to rotate about an RB-fixed axis, say $\hat{e}_3$, so that $\underline{\omega}_{m|I} = \omega\, \hat{e}_3$ and $\dot{\underline{\omega}}_{m|I} = \dot{\omega}\, \hat{e}_3$

Choose a fixed origin A at any point on the axis of rotation, which need not pass through the COM of RB

e.g.



The axes $\hat{e}_1$-$\hat{e}_2$-$\hat{e}_3$ may not necessarily be principal axes

$$[\underline{\omega}_{m|I}]_{\left(\begin{smallmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{smallmatrix}\right)} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \quad \text{or} \quad \underline{\omega}_{m|I} = \omega\, \hat{e}_3$$

and

$$[\dot{\underline{\omega}}_{m|I}]_{\left(\begin{smallmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{smallmatrix}\right)} = \begin{bmatrix} 0 \\ 0 \\ \dot{\omega} \end{bmatrix} \quad \text{or} \quad \dot{\underline{\omega}}_{m|I} = \dot{\omega}\, \hat{e}_3$$

Euler's 2nd axiom (or angular momentum balance) reduces to:

$$\begin{bmatrix} I^A_{11} & I^A_{12} & I^A_{13} \\ I^A_{21} & I^A_{22} & I^A_{23} \\ I^A_{31} & I^A_{32} & I^A_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \overset{\dot{\omega}}{\dot{\omega}_3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \overset{\omega}{\omega_3} \end{bmatrix} \times \left(\begin{bmatrix} I^A_{11} & I^A_{12} & I^A_{13} \\ I^A_{21} & I^A_{22} & I^A_{23} \\ I^A_{31} & I^A_{32} & I^A_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \overset{\omega}{\omega_3} \end{bmatrix} \right) = \begin{bmatrix} M_{A,1} \\ M_{A,2} \\ M_{A,3} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} I_{13}^A \\ I_{23}^A \\ I_{33}^A \end{bmatrix} \dot{\omega} + \begin{bmatrix} -I_{23}^A \\ I_{13}^A \\ 0 \end{bmatrix} \omega^2 = \begin{bmatrix} M_{A,1} \\ M_{A,2} \\ M_{A,3} \end{bmatrix}$$

$$\Rightarrow \underline{M}_A = \underbrace{\left(I_{13}^A \dot{\omega} - I_{23}^A \omega^2\right)}_{M_{A,1}} \hat{\underline{e}}_1 + \underbrace{\left(I_{23}^A \dot{\omega} + I_{13}^A \omega^2\right)}_{M_{A,2}} \hat{\underline{e}}_2 + \underbrace{I_{33}^A \dot{\omega}}_{M_{A,3}} \hat{\underline{e}}_3$$

The above equation holds true for $A \equiv$ COM, or for any A lying on the fixed axis of rotation (in this case $\hat{\underline{e}}_3$)

The axial component,

$$M_{A,3} = I_{33} \dot{\omega}(t)$$

relates the rotational motion to the externally applied torque $M_{A,3}$ about the fixed axis.

The remaining components

$$M_{A,1} = I_{13} \dot{\omega} - I_{23} \omega^2, \qquad M_{A,2} = I_{23} \dot{\omega} + I_{13} \omega^2$$

determine the reaction couples about the RB axes $\hat{\underline{e}}_1$ & $\hat{\underline{e}}_2$ required to control the rotation about $\hat{\underline{e}}_3$.

Now, also if $\hat{e}_3$ axis coincides with a principal axis at pt A,

then and only then:

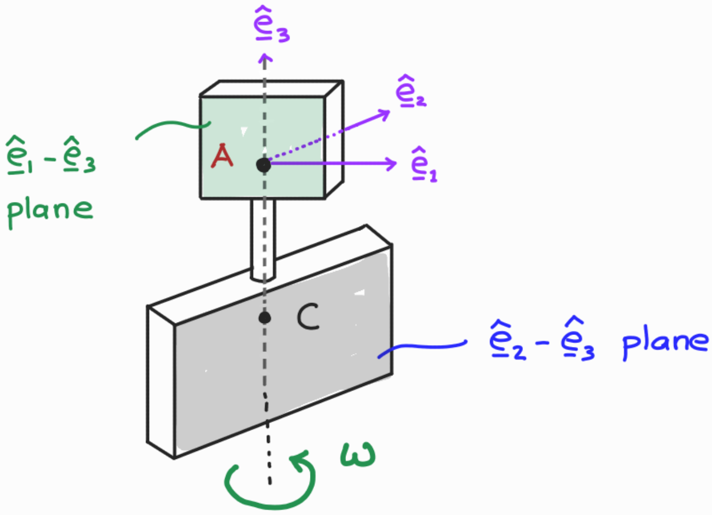
$$I^A_{13} = 0 \quad \text{and} \quad I^A_{23} = 0$$

Recall: $[\underline{\underline{I}}^A] = \begin{bmatrix} \checkmark & \checkmark & 0 \\ \checkmark & \checkmark & 0 \\ 0 & 0 & I^A_{33} \end{bmatrix}$ if $\hat{e}_3$ is principal axis dir.

$$\Rightarrow \quad M_{A,1} = 0 \quad \text{and} \quad M_{A,2} = 0$$

$$\text{and} \quad M_{A,3} = I^A_{33}\,\dot{\omega}$$

$$\Rightarrow \quad \underline{M}_A = I^A_{33}\,\dot{\omega}\,\hat{e}_3$$



body rotates about an RB-fixed axis $\hat{e}_3$, which also happens to be a principal axis in this case

Extra Material (if you are interested)

Definition of Inertial Frame

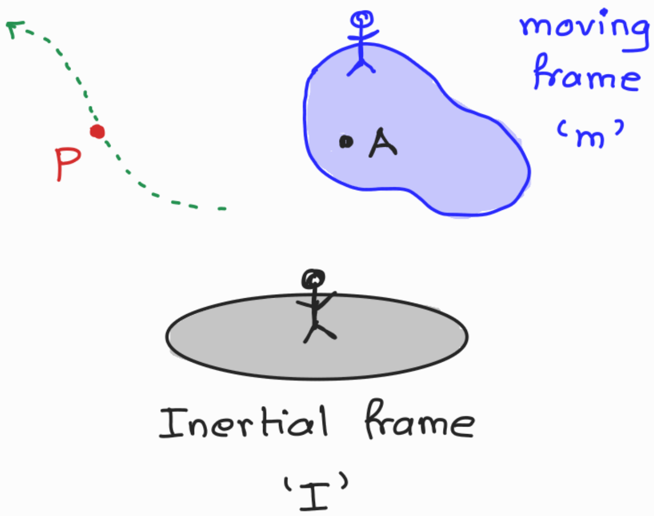
We defined inertial frame as a frame in which the two Euler's equations remain valid!

Is there a simpler definition on inertial frame? Yes

A frame in which a particle (chosen arbitrarily) experiences zero acceleration when the net force acting on it is zero!

Under what conditions will another frame 'm' be considered inertial relative to an already identified inertial frame 'I'?

We consider the motion of an arbitrary particle P w.r.t. the frames 'm' and 'I'



$$\underline{a}_{P|I} = \underline{a}_{P|m} + \underline{a}_{A|I} + \dot{\underline{\omega}}_{m|I} \times \underline{r}_{PA}$$

$$+ \underline{\omega}_{m|I} \times (\underline{\omega}_{m|I} \times \underline{r}_{PA})$$

$$+ 2\, \underline{\omega}_{m|I} \times \underline{v}_{P|m}$$

If the net force acting on the particle is zero in the inertial frame 'I', then from Euler's first equation

$$\underline{F}_R = \underline{0} \Rightarrow m\,\underline{a}_{P|I} = \underline{0} \Rightarrow \underline{a}_{P|I} = 0$$

Therefore,

$$\begin{aligned}\underline{a}_{P|I} = \underline{0} = \underline{a}_{P|m} + a_{A|I} &+ \dot{\underline{\omega}}_{m|I} \times \underline{r}_{PA} \\ &+ \underline{\omega}_{m|I} \times \left(\underline{\omega}_{m|I} \times \underline{r}_{PA}\right) \\ &+ 2\,\underline{\omega}_{m|I} \times \underline{v}_{P|m}\end{aligned}$$

P is arbitrarily chosen → $\underline{r}_{PA}$ is arbitrary

P is arbitrarily chosen → $\underline{v}_{P|m}$ is arbitrary

Further, if frame 'm' has to be inertial frame for P, then, by applying Euler's 1st eqn $m\,\underline{a}_{P|m} = \underline{F}_R = 0$

$$\Rightarrow \underline{a}_{P|m} = 0$$

$$\begin{aligned}\underline{0} = \cancelto{0}{\underline{a}_{P|m}} + a_{A|I} &+ \dot{\underline{\omega}}_{m|I} \times \overset{\neq 0}{\underline{r}_{PA}} \\ &+ \underline{\omega}_{m|I} \times \left(\underline{\omega}_{m|I} \times \overset{\neq 0}{\underline{r}_{PA}}\right) \\ &+ 2\,\underline{\omega}_{m|I} \times \overset{\neq 0}{\underline{v}_{P|m}}\end{aligned}$$

For the RHS to be zero, every single term must be zero (the reason being the arbitrariness of $\underline{r}_{PA}$ and $\underline{v}_{P|m}$), we should satisfy:

$$\underline{a}_{A|I} = \underline{0} \quad \Rightarrow$$

$$\dot{\underline{\omega}}_{m|I} \times \overset{\neq 0}{\underline{r}_{PA}} = \underline{0} \quad \Rightarrow \quad \dot{\underline{\omega}}_{m|I} = \underline{0}$$

$$\underline{\omega}_{m|I} \times \left(\underline{\omega}_{m|I} \times \overset{\neq 0}{\underline{r}_{PA}} \right) = \underline{0} \quad \Rightarrow \quad \underline{\omega}_{m|I} = 0$$

$$\therefore \quad \underbrace{\underline{\omega}_{m|I} = \underline{0} \quad , \quad \dot{\underline{\omega}}_{m|I} = \underline{0}}_{\text{No rotation at all}} \; , \quad \underbrace{\underline{a}_{A|F} = \underline{0}}_{\text{No linear acc.}}$$

Thus, the statement (that you are already familiar with)

A frame 'm' that is purely translating with constant velocity relative to an inertial frame is also inertial.