Recap

We have been talking about Euler's 2nd axiom in the last lecture, in particular, how Euler's 2nd axiom may be written about a reference point A which is moving (not fixed) w.r.t. an inertia frame 'I'. We derived that for such a moving point A, Euler's 2nd axiom turns out to be:

$$\boxed{\dot{\underline{H}}_{A|I} = \underline{M}_A - \underline{r}_{CA} \times m\, \underline{a}_{A|I}}$$

Alternative form of Euler's 2nd axiom

and that $\dot{\underline{H}}_{A|I} = \underline{M}_A$ is valid only for certain circumstances. In fact, if one chooses the point A to be the COM of the RB then

$$\dot{\underline{H}}_{C|I} = \underline{M}_C \quad \text{is valid!}$$

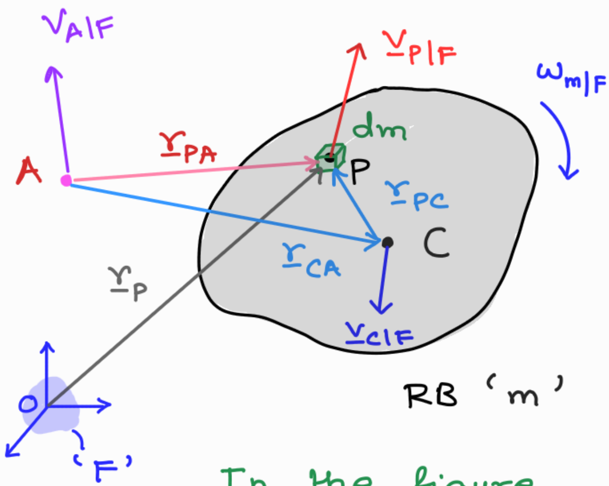
The motive behind trying to redefine Euler's 2nd axiom about a moving reference point will become more clear once we learn that the inertia tensor of an RB becomes a constant when calculated about a pt lying in the RB (or its massless extension). You see that any point lying on the RB need not be fixed to 'I' and therefore the need for alternate defn of Euler's 2nd axiom!

Can we develop a general equation for the angular momentum of a rigid body $\underline{H}_{A|F} = \underline{\quad ? \quad}$, much like linear momentum of an RB, $p|_F = m\,\underline{v}_{C|F}$?

Let the velocity of any point P of an RB relative to a reference point A ∈ RB (another material point in RB or a point on its massless extension) in a reference frame 'F' be denoted as:

$$\underline{v}_{PA|F} = \underline{v}_{P|F} - \underline{v}_{A|F}$$

could also be inertial frame 'I'



The angular momentum of the RB about A is:

$$\underline{H}_{A|F} = \int (\underline{r}_{PA} \times \underline{v}_{PA|F})\, dm$$

In the figure pt A is lying on a massless extension of the RB ⇒ dist of A from all pts of the RB is fixed.

Recall the velocity transfer relations:

(since P is a part of 'm')

$$\underline{v}_{P|F} = \cancel{\underline{v}_{P|m}}^{\,0} + \underline{v}_{A|F} + \underline{\omega}_{m|F} \times \underline{r}_{PA}$$

I.P. on 'm' (or massless extension)

$$\Rightarrow \quad \underline{v}_{P|F} - \underline{v}_{A|F} = \underline{\omega}_{m|F} \times \underline{r}_{PA}$$

$$\Rightarrow \quad \underline{v}_{PA|F} = \underline{\omega}_{m|F} \times \underline{r}_{PA}$$ ← use this relation in the defn of angular momentum

$$\underline{H}_{A|F} = \int \underline{r}_{PA} \times \left(\underline{\omega}_{m|F} \times \underline{r}_{PA}\right) dm$$

$$= \int \left\{ \left(\underline{r}_{PA} \cdot \underline{r}_{PA}\right) \underline{\omega}_{m|F} - \left(\underline{r}_{PA} \cdot \underline{\omega}_{m|F}\right) \underline{r}_{PA} \right\} dm$$

Let's write the vectors $\underline{r}_{PA}$ and $\underline{\omega}_{m|F}$ in terms of its compon- -ents in a certain csys $(\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3)$

$$[\underline{r}_{PA}]_{\begin{pmatrix}\hat{\underline{e}}_1\\ \hat{\underline{e}}_2\\ \hat{\underline{e}}_3\end{pmatrix}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad [\underline{\omega}_{m|F}]_{\begin{pmatrix}\hat{\underline{e}}_1\\ \hat{\underline{e}}_2\\ \hat{\underline{e}}_3\end{pmatrix}} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$[\underline{H}_{A|F}]_{\begin{pmatrix}\hat{\underline{e}}_1\\ \hat{\underline{e}}_2\\ \hat{\underline{e}}_3\end{pmatrix}} = \int \underbrace{\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)}_{x_1^2 + x_2^2 + x_3^2} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} - \underbrace{\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}\right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{let's do some manipulation}}$$

$$\underbrace{\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}\right)}_{\text{scalar}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}\right) = \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T\right) \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

(can be moved around freely)

$$= \left([\underline{r}_{PA}][\underline{r}_{PA}]^T\right)[\underline{\omega}_{m|F}]$$

$$= \underbrace{\left(\underline{r}_{PA} \otimes \underline{r}_{PA}\right)}_{\text{tensor product}} \underline{\omega}_{m|F}$$

$$\underline{H}_{A|F} = \int \left\{ \underbrace{\left(x_1^2 + x_2^2 + x_3^2\right)}_{r^2} \underline{\omega}_{m|F} - \left(\underline{r}_{PA} \otimes \underline{r}_{PA}\right) \underline{\omega}_{m|F} \right\} dm$$

$$= \left\{ \int \left[r^2 \, \underline{\underline{I}} - \left(\underline{r}_{PA} \otimes \underline{r}_{PA}\right) \right] dm \right\} \underline{\omega}_{m|F}$$

$$= \underbrace{\left\{ \int \left[\left(\underline{r}_{PA} \cdot \underline{r}_{PA}\right) \underline{\underline{I}} - \underline{r}_{PA} \otimes \underline{r}_{PA} \right] dm \right\}}_{} \underline{\omega}_{m|F}$$

$$\Downarrow$$

Inertia Tensor at pt A of the RB

(a 2nd order tensor)

$$\downarrow$$

two underlines for a 2nd order tensor $\longrightarrow$ $$\underline{\underline{I}}^A = \int \left[\left(\underline{r}_{PA} \cdot \underline{r}_{PA}\right) \underline{\underline{I}} - \underline{r}_{PA} \otimes \underline{r}_{PA} \right] dm$$

$$\therefore \quad \underline{H}_{A|F} = \underline{\underline{I}}^A \, \underline{\omega}_{m|F}$$

Expressed in matrix-vector form:

$$\underset{3\times 1}{\left[\underline{H}_{A|F}\right]} = \underset{3\times 3}{\left[\underline{\underline{I}}^A\right]} \; \underset{3\times 1}{\left[\underline{\omega}_{m|F}\right]}$$

representation of the inertia tensor in a csys is a matrix

Another way of deriving inertia tensor using Einstein's index notation is provided in the next page!

$$\underline{H}_{A|F} = \int \underline{r}_{PA} \times \left(\underline{\omega}_{m|F} \times \underline{r}_{PA} \right) dm$$

$$\left[\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\, \underline{b} - (a \cdot b)\, \underline{c} \right]$$

$$= \int \left[\left(\underline{r}_{PA} \cdot \underline{r}_{PA} \right) \underline{\omega}_{m|F} - \left(\underline{r}_{PA} \cdot \underline{\omega}_{m|F} \right) \underline{r}_{PA} \right] dm$$

Let $|\underline{r}_{PA}| = r$ (the magnitude of $\underline{r}_{PA}$)

$$\Rightarrow \quad \underline{r}_{PA} \cdot \underline{r}_{PA} = |\underline{r}_{PA}|^2 = r^2$$

and,

express $\underline{\omega}_{m|F}$ & $\underline{r}_{PA}$ in terms of its components expressed in (say) $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ csys.

$$\rightarrow \underline{\omega}_{m|F} = \omega_1 \hat{\underline{e}}_1 + \underline{\omega}_2 \hat{\underline{e}}_2 + \underline{\omega}_3 \hat{\underline{e}}_3 = \omega_i \hat{\underline{e}}_i$$

$$\rightarrow \underline{r}_{PA} = x_1 \hat{\underline{e}}_1 + x_2 \hat{\underline{e}}_2 + x_3 \hat{\underline{e}}_3 = x_i \hat{\underline{e}}_i$$

(index notation)

$$\therefore \quad \underline{r}_{PA} \cdot \underline{r}_{PA} = r^2 = x_1^2 + x_2^2 + x_3^2 = x_i^2$$

$$\underline{r}_{PA} \cdot \underline{}_{m|F} = x_1 \omega_1 + x_2 \omega_2 + x_3 \omega_3 = x_i \omega_i$$

Rewrite the expression of angular momentum $\underline{H}_{A|F}$:

$$\underline{H}_{A|F} = \int \left[r^2 \left(\omega_i \hat{\underline{e}}_i \right) - \left(x_j \omega_j \right) x_i \hat{\underline{e}}_i \right] dm$$

Compare $\Updownarrow$

$$\underline{H}_{A|F} = H_{A_1} \hat{\underline{e}}_1 + H_{A_2} \hat{\underline{e}}_2 + H_{A_3} \hat{\underline{e}}_3$$

$$H_{A_1} = \int r^2 \omega_1 - \left(x_j \omega_j \right) x_1 \, dm$$

$$H_{A_2} = \int r^2 \omega_2 - \left(x_j \omega_j \right) x_2 \, dm$$

$$H_{A_3} = \int r^2 \omega_3 - \left(x_j \omega_j \right) x_3 \, dm$$

$$\Rightarrow H_{A_i} = \int \left(r^2 \omega_i - \left(x_j \omega_j \right) x_i \right) dm$$

(index notation)

Recall Kronecker-Delta function

$$\delta_{ij} = \begin{cases} 1, & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}, \quad \text{also} \quad \delta_{ji} = \delta_{ij}$$

(symmetric)

Let us write ω_1, ω_2, and ω_3 in terms of δ_{ij}

$$\omega_1 = \cancelto{1}{\delta_{11}}\,\omega_1 + \cancelto{0}{\delta_{12}}\,\omega_2 + \cancelto{0}{\delta_{13}}\,\omega_3 = \delta_{1j}\,\omega_j$$

Similarly, $\omega_2 = \delta_{2j}\,\omega_j$, and $\omega_3 = \delta_{3j}\,\omega_j$

$$\Rightarrow \quad \omega_i = \delta_{ij}\,\omega_j \qquad \text{(compact index notation)}$$

Rewrite the components $\underline{H}_{A_i}$ in INDEX notation

$$\underline{H}_{A_i} = \int \left[r^2 \underline{\underline{\omega_i}} - (x_j\,\omega_j)\,x_i \right] dm$$

$$= \int \left[r^2 \delta_{ij}\,\underline{\underline{\omega_j}} - (x_j\,\underline{\underline{\omega_j}})\,x_i \right] dm$$

$$H_{A_i} = \left[\int \left(r^2 \delta_{ij} - x_i\,x_j \right) dm \right] \underline{\underline{\omega_j}}$$

constant for integration

Note that the RHS of H_{A_i} is a summation of three terms

$j = 1, 2, 3$

(because 'j' appears twice)

$$H_{A_i} = \left[\int \left(r^2 \delta_{i\underline{\underline{j}}} - x_i\,x_{\underline{\underline{j}}} \right) dm \right] \omega_{\underline{\underline{j}}}$$

$$H_{A_i} = \left[\int \left(r^2 \delta_{i1} - x_i x_1\right) dm\right] \omega_1 + \left[\int \left(r^2 \delta_{i2} - x_i x_2\right) dm\right] \omega_2$$

$$+ \left[\int \left(r^2 \delta_{i3} - x_i x_3\right) dm\right] \omega_3$$

'A' is the pt abt which angular momentum is calculated

Define $\boxed{I_{ij}^{A} \equiv \int \left(r^2 \delta_{ij} - x_i x_j\right) dm}$

and we can write H_{A_i} in terms of this new symbol I_{ij}^{A}

$$H_{A_i} = I_{\underline{\underline{j}}}^{A} \, \omega_{\underline{\underline{j}}} \quad \Rightarrow \quad \begin{aligned} H_{A_1} &= I_{11}^A \omega_1 + I_{12}^A \omega_2 + I_{13}^A \omega_3 \\ H_{A_2} &= I_{21}^A \omega_1 + I_{22}^A \omega_2 + I_{23}^A \omega_3 \\ H_{A_3} &= I_{31}^A \omega_1 + I_{32}^A \omega_2 + I_{33}^A \omega_3 \end{aligned}$$

summation index

In matrix-vector form,

$$\underbrace{\begin{bmatrix} H_{A_1} \\ H_{A_2} \\ H_{A_3} \end{bmatrix}}_{\underline{H}_{A|F}} = \underbrace{\begin{bmatrix} I_{11}^A & I_{12}^A & I_{13}^A \\ I_{21}^A & I_{22}^A & I_{23}^A \\ I_{31}^A & I_{32}^A & I_{33}^A \end{bmatrix}}_{\underline{\underline{I}}^A} \underbrace{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}}_{\underline{\omega}_{m|F}}$$

Much like $\underline{p}|_F = m \underline{v}_{c|F}$

$$\Rightarrow \quad \underline{H}_{A|F} = \boxed{\underline{\underline{I}}^A} \, \underline{\omega}_{m|F}$$

SECOND-ORDER TENSOR

↳ has a 3×3 matrix representation $\left[\underline{\underline{I}}^A\right]$

vectors (3×1)

This second order tensor $\underline{\underline{I}}^A$ is also called Inertia tensor of the RB at point A of the RB

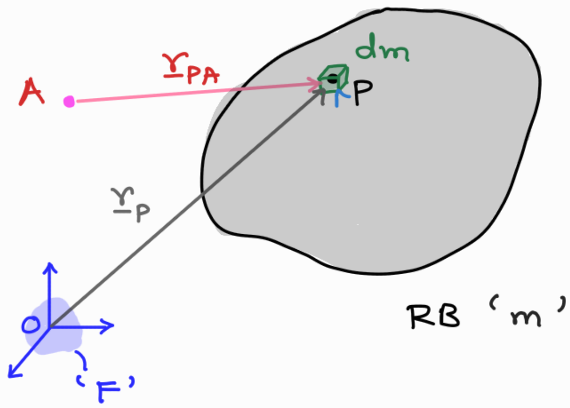
Definition: Inertia Tensor $\underline{\underline{I}}^A$ about point A on the RB

If $\underline{r}_{PA}$ denotes the position vector from a reference point A of an RB (or massless extension of RB) to a small material point of mass dm at point P, then the tensor $\underline{\underline{I}}^A$ defined by:

tensor product

$$\underline{\underline{I}}^A = \int \left[(\underline{r}_{PA} \cdot \underline{r}_{PA}) \underline{\underline{I}} - \underline{r}_{PA} \otimes \underline{r}_{PA} \right] dm$$

is called the INERTIA TENSOR at point A



$\underline{\underline{I}} \leftarrow$ Identity tensor

↳ matrix representation is an identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Tensor product of two vectors

$$\underline{a} \otimes \underline{b} = \underline{a}\, \underline{b}^T = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

$\Downarrow$

Using the result

$\downarrow$

$$\underline{r}_{PA} \otimes \underline{r}_{PA} = \underline{r}_{PA}\, \underline{r}_{PA}^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} x_1 x_1 & x_1 x_2 & x_1 x_3 \\ x_2 x_1 & x_2 x_2 & x_2 x_3 \\ x_3 x_1 & x_3 x_2 & x_3 x_3 \end{bmatrix}$$

Some properties of the Inertia Tensor

(1) In general, $\underline{\underline{I}}^A \neq \underline{\underline{I}}^B$, that is inertia tensor will depend on the choice of reference point A or B on the rigid body.

(2) Is the inertia tensor symmetric? YES!

The way of proof will be that the representation of the inertia tensor in a csys, which is a matrix $[\underline{\underline{I}}^A]_{3\times3}$ shall be proved to be symmetric.

$$\underline{r}_{PA} = x_1\,\hat{\underline{e}}_1 + x_2\,\hat{\underline{e}}_2 + x_3\,\hat{\underline{e}}_3\,, \quad \text{or} \quad [\underline{r}_{PA}]_{(\hat{\underline{e}}_1\,\hat{\underline{e}}_2\,\hat{\underline{e}}_3)} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Tensor product

$$\underline{\underline{I}}^A = \int \left[(\underline{r}_{PA} \cdot \underline{r}_{PA})\,\underline{\underline{I}} - \underline{r}_{PA} \otimes \underline{r}_{PA} \right] dm$$

$$\Rightarrow \underset{3\times3}{[\underline{\underline{I}}^A]} = \int \left\{ \underbrace{\left(\underset{1\times3}{[\underline{r}_{PA}]^T}\, \underset{3\times1}{[\underline{r}_{PA}]} \right)}_{1\times1} \underset{3\times3}{[\underline{\underline{I}}]} - \underbrace{\underset{3\times1}{[\underline{r}_{PA}]}\, \underset{1\times3}{[\underline{r}_{PA}]^T}}_{3\times3} \right\} dm$$

$$[\underline{\underline{I}}^A] = \int \left\{ \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \right\} dm$$

$$[\underline{\underline{I}}^A]^T = \int \left\{ \underbrace{\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)}_{\text{scalar}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T - \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \right)^T \right\} dm$$

(transpose of a scalar is the same scalar)

$$(\underline{a}\, \underline{b}^T)^T = \underline{b}\, \underline{a}^T$$

$$= \int \left\{ \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \right\} dm$$

$$= [\underline{\underline{I}}^A]$$

$$\therefore \quad [\underline{\underline{I}}^A]^T = [\underline{\underline{I}}^A] \Rightarrow \text{Inertia matrix is Symmetric}$$

Since the representation of the inertia tensor is symmetric, the inertia tensor itself is also symmetric

Another way of proof using index notation

$$I_{ij}^{A} = \int (r^2 \delta_{ij} - x_i x_j)\, dm$$

Interchange the indices $i \leftrightarrow j$ and the value does not change.

$$I_{ji}^{A} = \int (r^2 \underline{\underline{\delta_{ji}}} - x_j x_i)\, dm$$

$$= \int (r^2 \delta_{ij} - x_j x_i)\, dm \qquad [\because \delta_{ji} = \delta_{ij}]$$

$$= I_{ij}^{A}$$

$$\therefore \quad I_{ij}^{A} = I_{ji}^{A}$$

$$\Rightarrow \quad [\underline{\underline{I}}^{A}] \text{ is a symmetric matrix}$$

3 Note the reference point A need to be fixed in the RB (or its massless extension) and therefore lies on the reference frame 'm' of the RB, and is not fixed to the frame 'F'.

Because A is fixed to the RB, $\underline{\underline{I}}^{A}$ is a constant tensor as $\underline{r}_{PA}$ will remain constant.

$\Downarrow$ Great advantage for calculations

If pt 'A' was fixed to frame 'F', then $\underline{r}_{PA}$ would have varied with time due to RB's motion and $\underline{\underline{I}}^{A}$ would not be constant.

④ The components of the matrix representation of $\underline{\underline{I}}^A$ depends upon the choice of orientation of the coordinate system. But the inertia tensor itself does not change with the choice of the csys! (Much like how vectors do not depend on the choice of csys, however, their components depend on the orientation of the csys)

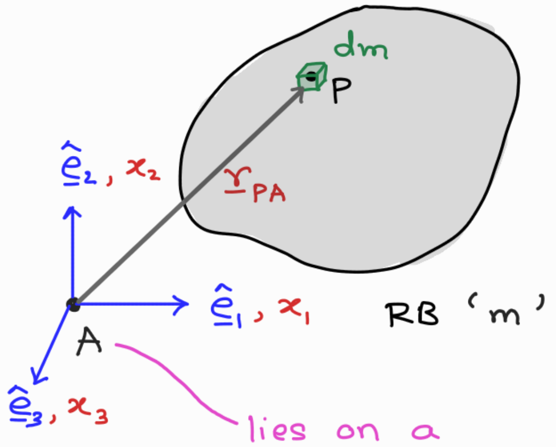
⑤ Has dimensions (mass) (distance)^2

and are expressed in the units kg m^2

Physical Significance of Inertia Tensor

The inertia tensor is involved in the analysis of any body having angular (or rotational) acceleration. Just as the mass of a body is a measure of the resistance to translational acceleration, the inertia tensor $\underline{\underline{I}}^A$ is a measure of resistance to rotational acceleration of the body

Cartesian components of $[\underline{\underline{I}}^A]$



lies on a massless extension of RB 'm'

$$\underline{r}_{PA} = x_1\,\hat{\underline{e}}_1 + x_2\,\hat{\underline{e}}_2 + x_3\,\hat{\underline{e}}_3$$

$$\underline{r}_{PA}\cdot\underline{r}_{PA} = r^2 = x_1^2 + x_2^2 + x_3^2$$

$$\boxed{I^A_{ij} = \int \left(r^2\,\delta_{ij} - x_i\,x_j\right) dm}$$

Set $i=1$, $j=1$ $\quad I^A_{11} = \int \left(\cancel{x_1^2} + x_2^2 + x_3^2 - \cancel{x_1 x_1}\right) dm$

$$= \int \left(x_2^2 + x_3^2\right) dm$$

Similarly,

$$I^A_{22} = \int \left(x_1^2 + x_3^2\right) dm, \quad \text{and} \quad I^A_{33} = \int \left(x_1^2 + x_2^2\right) dm$$

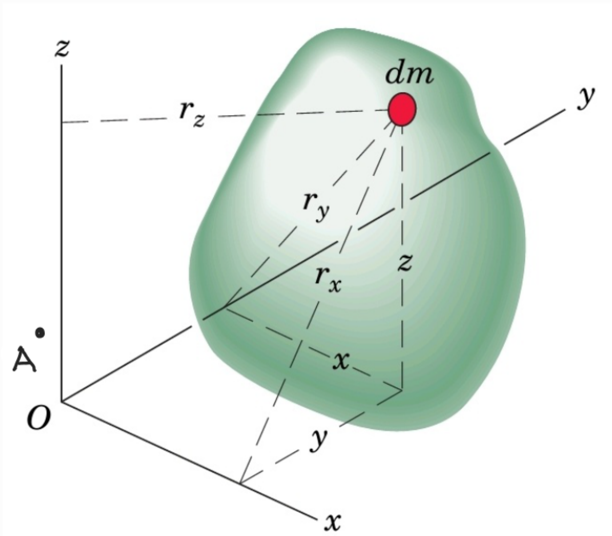
These three components are called moments of inertia

$$\left.\begin{aligned} I^A_{11} &= \int \left(x_2^2 + x_3^2\right) dm \\ I^A_{22} &= \int \left(x_1^2 + x_3^2\right) dm \\ I^A_{33} &= \int \left(x_1^2 + x_2^2\right) dm \end{aligned}\right\}$$

Moments of inertia at A

(w.r.t. $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ csys)

$$\boxed{\begin{aligned} I^A_{xx} &= \int r_x^{\,2}\,dm = \int (y^2 + z^2)\,dm \\ I^A_{yy} &= \int r_y^{\,2}\,dm = \int (z^2 + x^2)\,dm \\ I^A_{zz} &= \int r_z^{\,2}\,dm = \int (x^2 + y^2)\,dm \end{aligned}}$$



The rest six components of the inertia tensor are called products of inertia

For $i \neq j$

$$I_{12}^{A} = I_{21}^{A} = \int \left(r^2 \cancel{\delta_{12}}^{\,0} - x_1 x_2 \right) dm$$

$$\Rightarrow I_{12}^{A} = I_{21}^{A} = -\int x_1 x_2 \, dm$$

Therefore,

$$\left.\begin{aligned} I_{12}^{A} &= I_{21}^{A} = -\int x_1 x_2 \, dm \\ I_{13}^{A} &= I_{31}^{A} = -\int x_1 x_3 \, dm \\ I_{23}^{A} &= I_{32}^{A} = -\int x_2 x_3 \, dm \end{aligned}\right]$$

Products of inertia at A
(w.r.t. $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ csys)

* While moments of inertia are positive quantities, products of inertia may be either positive or negative

Note that in many dynamics books the products of inertia are defined somewhat differently:

$$\boxed{\begin{aligned} I_{xy}^{A} &= I_{yx}^{A} = \int xy \, dm \\ I_{xz}^{A} &= I_{zx}^{A} = \int xz \, dm \\ I_{yz}^{A} &= I_{zy}^{A} = \int yz \, dm \end{aligned}} \quad \Leftarrow \quad \begin{aligned} I_{xy}^{A} &= -I_{12}^{A} \\ I_{xz}^{A} &= -I_{13}^{A} \\ I_{yz}^{A} &= -I_{23}^{A} \end{aligned}$$

* Please exercise caution when consulting other sources!

For an RB with uniform mass density, ρ = constant may be extracted from the inertia integrals, and the integral will remain as volume integrals.

$$I^A_{ij} = \int (r^2 \delta_{ij} - x_i x_j)\, \rho \, dV$$

$$= \rho \int (r^2 \delta_{ij} - x_i x_j)\, dV$$

Note, once again, that the values of the components depend upon the choice of point A and the orientation of the csys.

$\underline{\underline{I}}^A \rightarrow$ Inertia Tensor of RB at point A

(does not depend on the orientation of csys)

$[\underline{\underline{I}}^A] \rightarrow$ Inertia Matrix of RB at point A w.r.t csys $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$

$$[\underline{\underline{I}}^A] = \begin{bmatrix} I^A_{11} & I^A_{12} & I^A_{13} \\ I^A_{21} & I^A_{22} & I^A_{23} \\ I^A_{31} & I^A_{32} & I^A_{33} \end{bmatrix} = \begin{bmatrix} I^A_{xx} & -I^A_{xy} & -I^A_{xz} \\ -I^A_{yx} & I^A_{yy} & -I^A_{yz} \\ -I^A_{zx} & -I^A_{zy} & I^A_{zz} \end{bmatrix}$$

(in some other books)

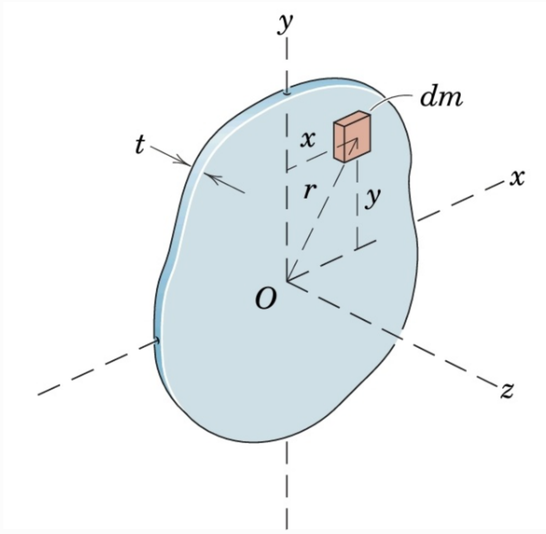
components depend on the orientation of csys

Moment of Inertia for a Thin Plane Body

For a thin, plane flat body lying in the x-y plane of negligible thickness 't',

$z \to 0$

and the components of inertia tensor for the thin plane body are:



$$I_{11}^{A} = \int (y^2 + \cancel{z^2}^{\,0})\, dm = \int y^2\, dm$$

$$I_{22}^{A} = \int (x^2 + \cancel{z^2}^{\,0})\, dm = \int x^2\, dm$$

$$I_{33}^{A} = \int (x^2 + y^2)\, dm = I_{11}^{A} + I_{22}^{A}$$

this relationship $I_{33}^{A} = I_{11}^{A} + I_{22}^{A}$ is called the perpendicular axes theorem

$$I_{zz}^{A} = I_{xx}^{A} + I_{yy}^{A}$$

$$I_{12}^{A} = I_{21}^{A} = -\int xy\, dm$$

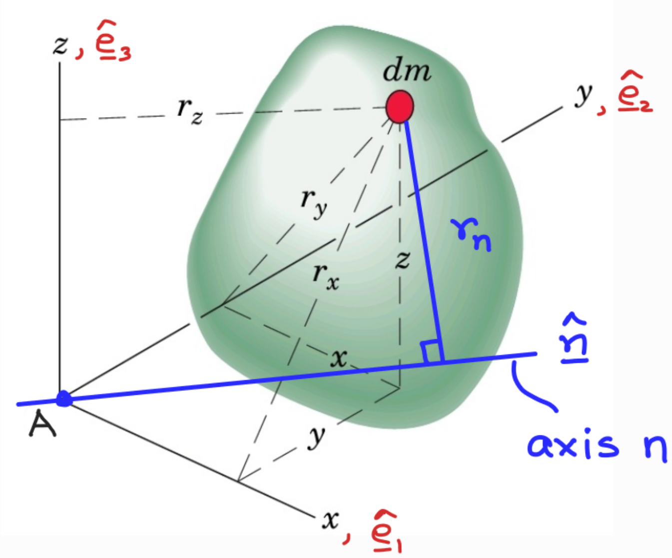
$$I_{13}^{A} = I_{31}^{A} = -\int x\cancel{z}^{\,0}\, dm = 0$$

$$I_{23}^{A} = I_{32}^{A} = -\int y\cancel{z}^{\,0}\, dm = 0$$

Moment of Inertia about an Arbitrary Axis

The moment of inertia components of the inertia tensor have the general form

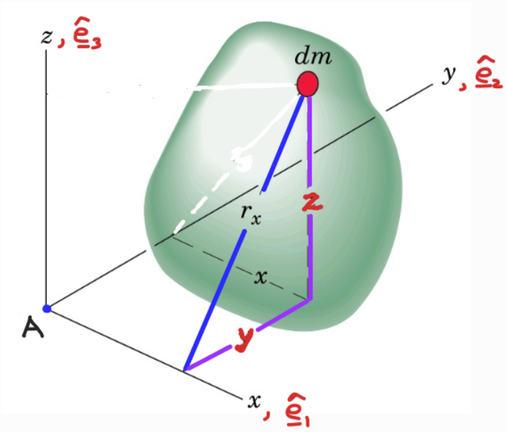
$$I_{nn} = \int r_n^2 \, dm$$



$r_n \leftarrow$ perpendicular distance from axis n (with unit vector $\hat{n}$) to the element of mass dm

Thus, $I_{nn} = \int r_n^2 \, dm$ is called the moment of inertia about the axis $\underline{n}$ through the point A.

Ex. Moment of inertia abt $\underline{e}_1$-axis through A, $\hat{n} = \hat{e}_1$



$$I_{11}^A = I_{xx}^A = \int r_x^2 \, dm$$

$$= \int (y^2 + z^2) \, dm$$

Radius of $\underset{\sim}{\text{Gyration}}$

meaning rotation

The radius of gyration about an axis $\underline{n}$ is a positive scalar k_n^A defined as:

moment of inertia about an axis $\underline{n}$ passing through pt A

$$k_n^A = \sqrt{\frac{I_{nn}^A}{m}}$$

total mass of RB

Ex:

$$k_x^A = \sqrt{\frac{I_{xx}^A}{m}} \quad \text{or} \quad k_1^A = \sqrt{\frac{I_{11}^A}{m}}$$

Similarly,

$$k_2^A = \sqrt{\frac{I_{22}^A}{m}} \quad , \quad k_3^A = \sqrt{\frac{I_{33}^A}{m}}$$

* k_n^A is a measure of the distribution of mass of a given body about the axis $\underline{n}$.

* Physically, it means that if all the mass of the body could be concentrated at a distance k_n^A from axis $\underline{n}$, the moment of inertia I_{nn}^A would remain unchanged.

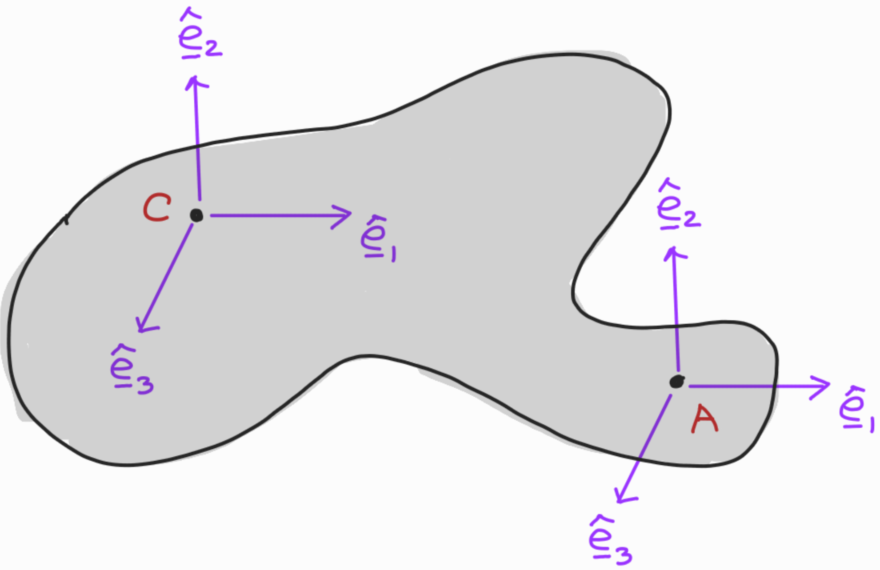
Parallel Axes Theorem

This theorem is a useful transformation rule that relates the inertia matrix components computed at a reference point A on the body to the components of inertia matrix at center of mass C of the RB in parallel csys

Consider two parallel csys axes, one at COM C and one at A

Consider that $[\underline{\underline{I}}^C]$ is known w.r.t. $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$ csys

components of inertia matrix at the COM C



$[\underline{\underline{I}}^A] \leftarrow$ inertia matrix at point A (w.r.t $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$ csys)

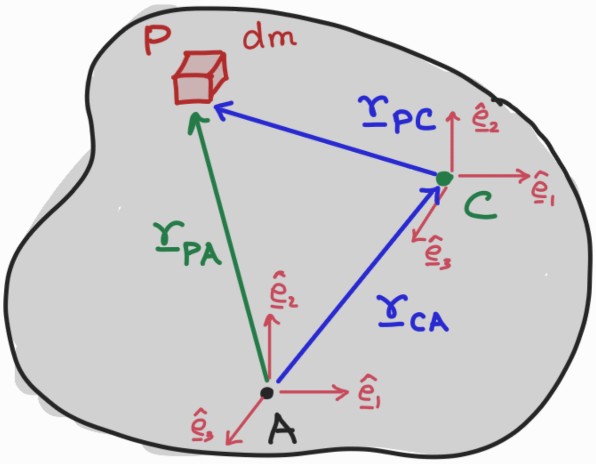
How is $[\underline{\underline{I}}^A]_{\begin{bmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{bmatrix}}$ related to $[\underline{\underline{I}}^C]_{\begin{bmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{bmatrix}}$?

$$\underline{r}_{PA} = x_1\,\hat{\underline{e}}_1 + x_2\,\hat{\underline{e}}_2 + x_3\,\hat{\underline{e}}_3 = x_i\,\hat{\underline{e}}_i$$

$$\underline{r}_{CA} = x_{c_1}\,\hat{\underline{e}}_1 + x_{c_2}\,\hat{\underline{e}}_2 + x_{c_3}\,\hat{\underline{e}}_3 = x_{c_i}\,\hat{\underline{e}}_i \quad \text{[Constant vector]}$$

$$\underline{r}_{PC} = \bar{x}_1\,\hat{\underline{e}}_1 + \bar{x}_2\,\hat{\underline{e}}_2 + \bar{x}_3\,\hat{\underline{e}}_3 = \bar{x}_i\,\hat{\underline{e}}_i$$

Consider the COM C as the origin



Moment of inertia terms

e.g.

$$I_{11}^A = \int (x_2^2 + x_3^2)\, dm$$

$$= \int \left[(x_{c_2} + \bar{x}_2)^2 + (x_{c_3} + \bar{x}_3)^2 \right] dm$$

$$= \int (\bar{x}_2^2 + \bar{x}_3^2) + \int \boxed{(x_{c_2}^2 + x_{c_3}^2)}\, dm$$

const. w.r.t integration

$$+ 2\int \boxed{x_{c_2}}\,\bar{x}_2\, dm + 2\int \boxed{x_{c_3}}\,\bar{x}_3\, dm$$

$$= \underbrace{\int (\bar{x}_2^2 + \bar{x}_3^2)\, dm}_{I_{11}^C} + (x_{c_2}^2 + x_{c_3}^2)\Big(\cancelto{m}{\int dm}\Big)$$

$$+ 2\,x_{c_2}\Big(\cancelto{0}{\int \bar{x}_2\, dm}\Big) + 2\,x_{c_3}\Big(\cancelto{0}{\int \bar{x}_3\, dm}\Big)$$

$\int \bar{x}_i \, dm = 0$ since $\underline{r}_c = \dfrac{\int \underline{r}_p \, dm}{\int dm}$ is the coordinates of the COM and we have already assumed

$\underline{r}_c = \underline{0}$ (origin)

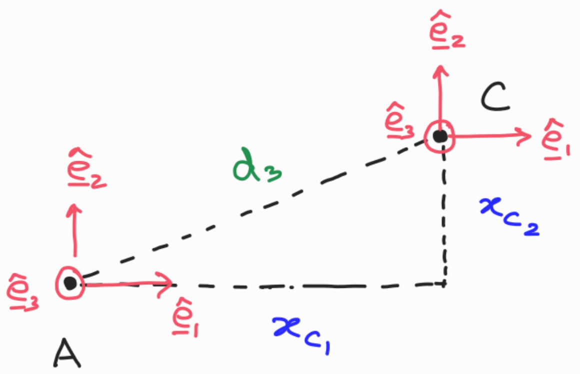
$$I_{11}^A = I_{11}^C + m \underbrace{(x_{c_2}^2 + x_{c_3}^2)}_{d_1^2}$$

Similarly,

$$I_{22}^A = I_{22}^C + m \underbrace{(x_{c_1}^2 + x_{c_3}^2)}_{d_2^2}$$

$$I_{33}^A = I_{33}^C + m \underbrace{(x_{c_1}^2 + x_{c_2}^2)}_{d_3^2}$$

Note: Moments of inertia have their smallest values at COM 'C' compared to a parallel axis at any other point



$\Rightarrow \quad d_3^2 = x_{c_1}^2 + x_{c_2}^2$

d_3 is the perpendicular distance between $\hat{e}_3$-axis at A and that at C

Similarly

$d_1^2 = x_{c_2}^2 + x_{c_3}^2$

$d_2^2 = x_{c_1}^2 + x_{c_3}^2$

Similarly, products of inertia terms:

$$I_{12}^{A} = -\int x_1 \, x_2 \, dm = -\int (x_{c_1} + \bar{x}_1)(x_{c_2} + \bar{x}_2) \, dm$$

$$\Rightarrow I_{12}^{A} = I_{12}^{C} - m \, x_{c_1} x_{c_2}$$ (Verify at home)

and, $I_{23}^{A} = I_{23}^{C} - m \, x_{c_2} x_{c_3}$,

and, $I_{13}^{A} = I_{13}^{C} - m \, x_{c_1} x_{c_3}$

Notes

1> Parallel axes theorem is NOT valid for any two general points. One of the points must be the COM C,

2> For any two general points A and B:

$[\underline{\underline{I}}^{A}]$ is known, find $[\underline{\underline{I}}^{B}]$?

Do this: $[\underline{\underline{I}}^{A}] \xrightarrow[\text{theorem}]{\text{//el axes}} [\underline{\underline{I}}_{c}] \xrightarrow[\text{theorem}]{\text{//el axes}} [\underline{\underline{I}}^{B}]$