Recap

We had introduced Euler's axioms where we found that we needed to find the net external forces and net ext. moments acting on a system. Towards this goal, we had introduced the idea of equivalent force systems, using which we reduced the effect of several forces and couples acting on a system to an equivalent single resultant force and a single resultant couple, or an equivalent wrench.

Before we find the equivalent force system, it is needed that we identify all the forces and couples that are exerted on the system — the external forces and couples. In the last lecture, we started discussing how to draw FBDs where we identify all contact and body forces and contact moments acting externally to the system under consideration.

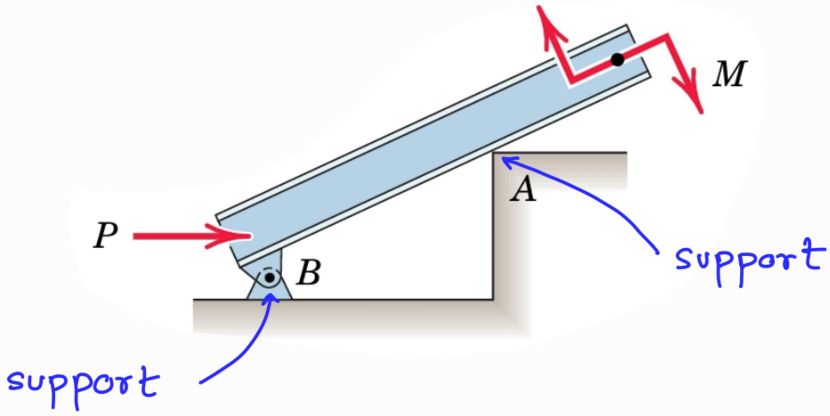
In this lecture, we will look at a class of ext. contact forces & moments that are generated (than directly applied) when a system's movement is constrained by some supports.

→ called reactions

Reactions at Supports

In many situations, the motion of an RB or a system is constrained by some supports

Ex:

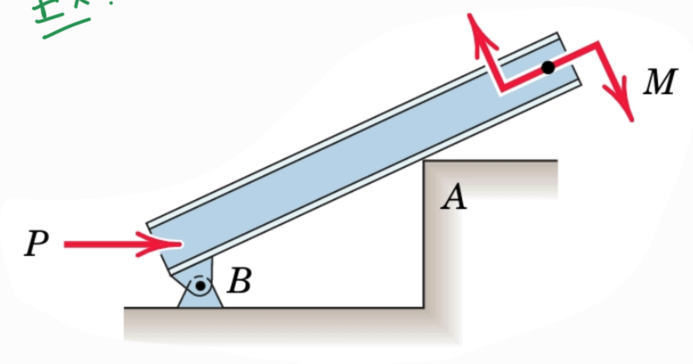


When a force is applied on the system, the support generates a force system (force and moment/couple)

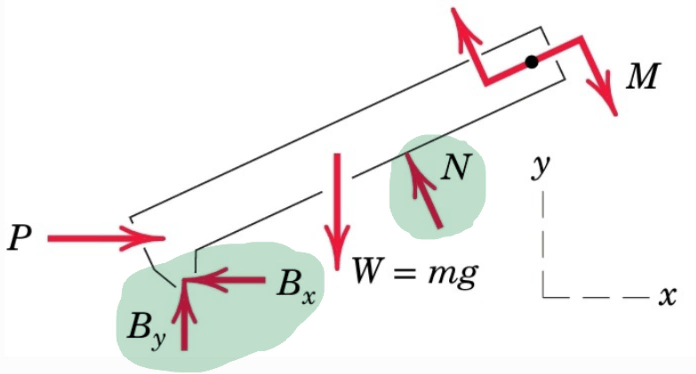
contact force & couple
The surface force system that develop at the supports or points of contact on the body of interest is called the reaction force system (in short REACTIONS)

While drawing an FBD of a system, we should remove the physical details of the support and replace those details by a reaction force system!

Ex:

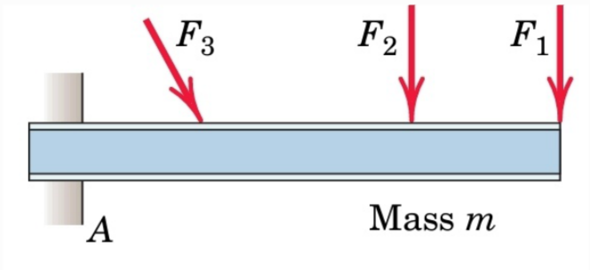


⇒

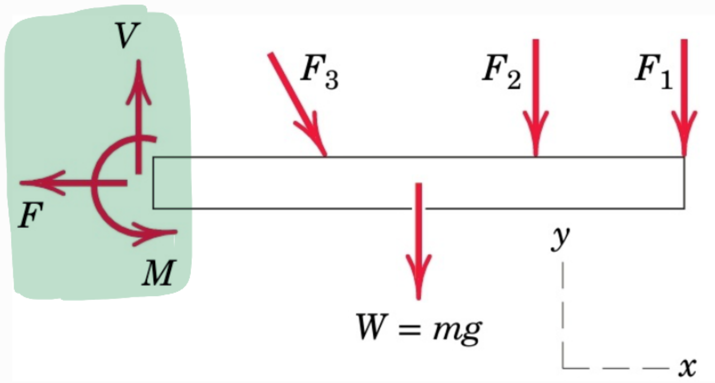


FBD

General Rule : If the support prevents translation in a given direction, then a force must be developed on the system in that direction. Likewise, if rotation is prevented, a couple moment must be exerted on the system.

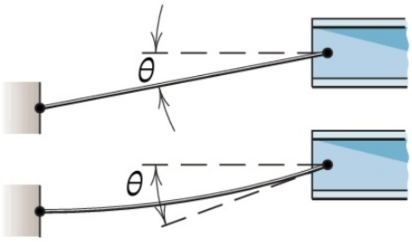
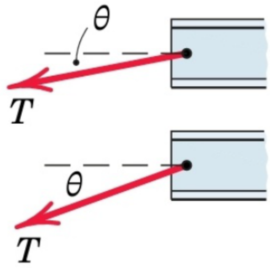
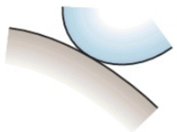
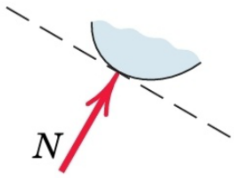
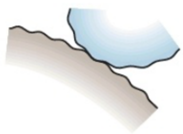
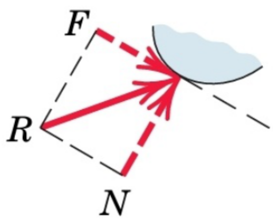
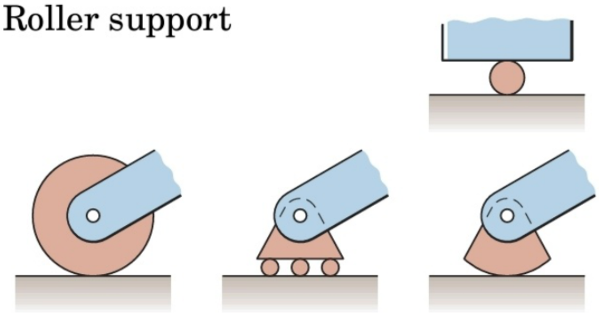
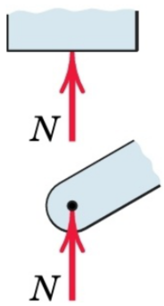
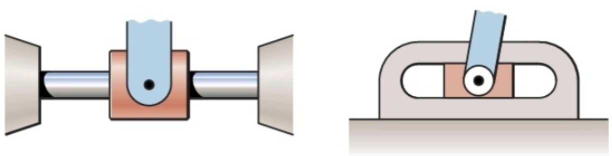
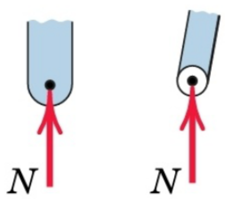


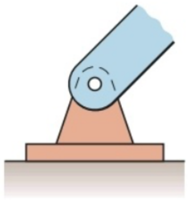
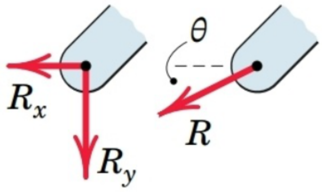
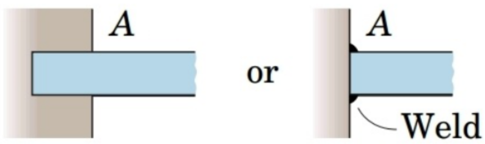
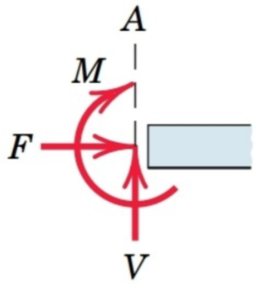
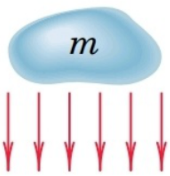
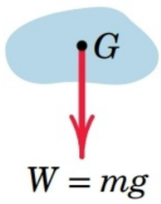
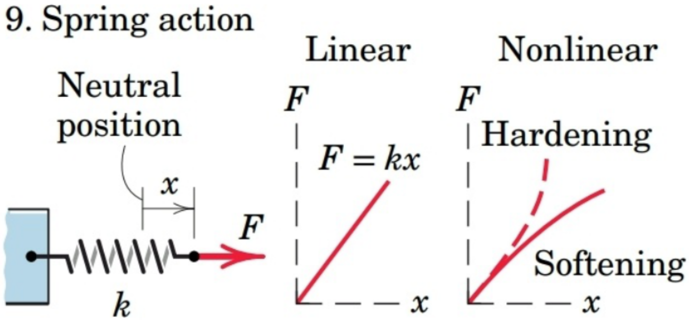
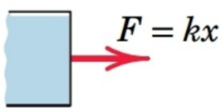
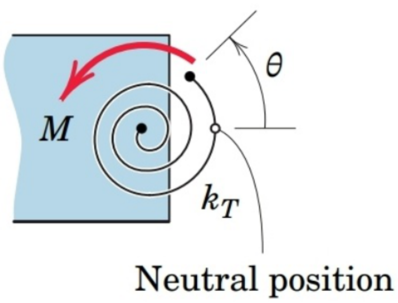
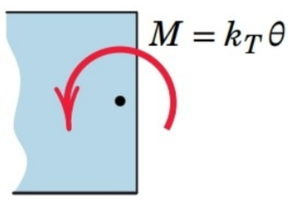
⇒



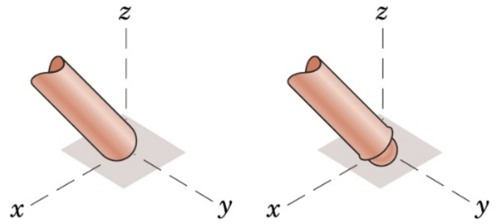
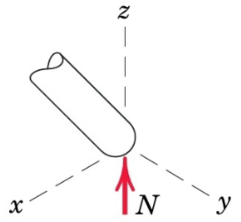
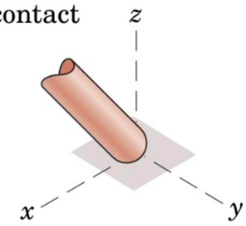
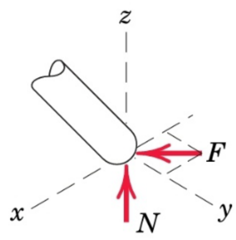
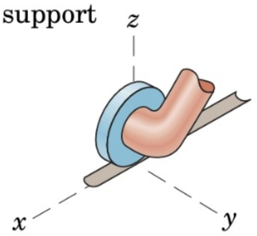
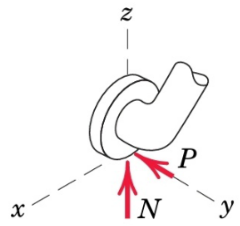
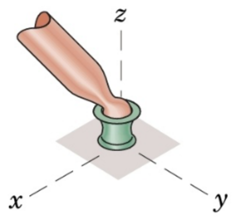
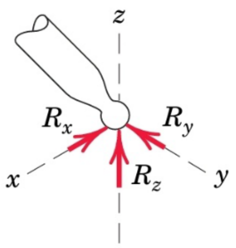
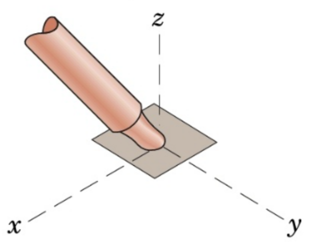
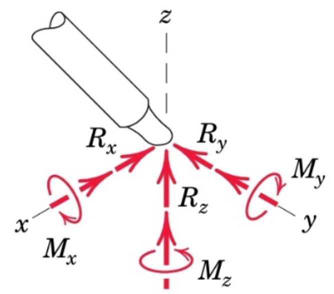
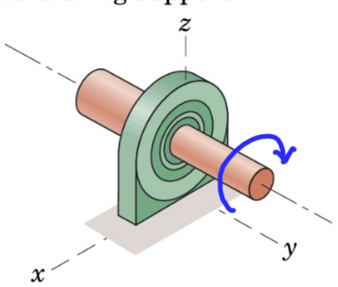
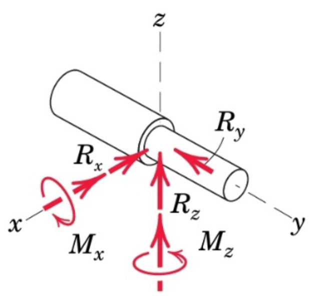
Let us consider few cases of different support systems in 2D first and examine the reactions generated.

Support reactions for 2D analysis

Type of Contact and Force Origin	Action on Body to Be Isolated
1. Flexible cable, belt, chain, or rope Weight of cable negligible Weight of cable not negligible 	 Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.
2. Smooth surfaces 	 Contact force is compressive and is normal to the surface.
3. Rough surfaces 	 Rough surfaces are capable of supporting a tangential component *F* (frictional force) as well as a normal component *N* of the resultant contact force *R*.
4. Roller support 	 Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.
5. Freely sliding guide 	 Collar or slider free to move along smooth guides; can support force normal to guide only.

Type of Contact and Force Origin	Action on Body to Be Isolated	
6. Pin connection 		A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. We may either show two components R_x and R_y or a magnitude R and direction θ.
7. Built-in or fixed support 		A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.
8. Gravitational attraction Not a reaction! 		The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center of gravity G.
9. Spring action 		Spring force is tensile if the spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.
10. Torsional spring action 		For a linear torsional spring, the applied moment M is proportional to the angular deflection θ from the neutral position. The stiffness k_T is the moment required to deform the spring one radian.

Reaction force systems in 3D analysis

Type of Contact and Force Origin	Action on Body to Be Isolated (Unknowns)	
1. Member in contact with smooth surface, or ball-supported member 		Force must be normal to the surface and directed toward the member.
2. Member in contact with rough surface 		The possibility exists for a force F tangent to the surface (friction force) to act on the member, as well as a normal force N.
3. Roller or wheel support with lateral constraint 		A lateral force P exerted by the guide on the wheel can exist, in addition to the normal force N.
4. Ball-and-socket joint 		A ball-and-socket joint free to pivot about the center of the ball can support a force **R** with all three components.
5. Fixed connection (embedded or welded) 		In addition to three components of force, a fixed connection can support a couple **M** represented by its three components.
6. Thrust-bearing support 		Thrust bearing is capable of supporting axial force R_y as well as radial forces R_x and R_z. Couples M_x and M_z

In a 3D case, you may have a maximum of SIX unknown reaction force system (3 forces + 3 moment couples)

We have now learned

a) how to draw an FBD of a system by tracking all contact forces and body forces,

applied   reactions

b) how to replace a force system by an equivalent resultant force system at a given point

These will be useful when we deal with Euler's axioms

Recall the two axioms:

1st axiom:

$$\dot{\underline{p}}\big|_I = \underline{F}_R$$

(Rate of change of linear momentum = net external force)

$$\Rightarrow \quad m\, \underline{a}_{c|I} = \underline{F}_R$$

center of mass (COM)

2nd axiom:

$$\dot{\underline{H}}_{O|I} = \underline{M}_O$$

(Rate of change of angular momentum = net external moment)

point O must be a point fixed in the inertial reference frame I !!

[Note: Forces & moments are frame-invariant]

These two sets of relations leads to the equations of motion which are effectively six scalar equations (equivalent to two vector equations of motion). Together with appropriate initial conditions, these are used to determine the general motion of an RB.

Determination of this motion, however, does not always split neatly into a translational part and a rotational part. Infact in many cases, these two parts are coupled together that give rise to strongly coupled system of nonlinear ODEs

Nonlinear ODE : $$\frac{d\underline{x}(t)}{dt} = \underline{g}\left(\underline{x}(t), \underline{F}_R(t), \underline{M}_O(t)\right)$$

Initial condition : $$\underline{x}(t=0) = \underline{x}_0$$

$$\underline{x} = \begin{bmatrix} \underline{x}_{tran} \\ \underline{x}_{rot} \end{bmatrix}$$

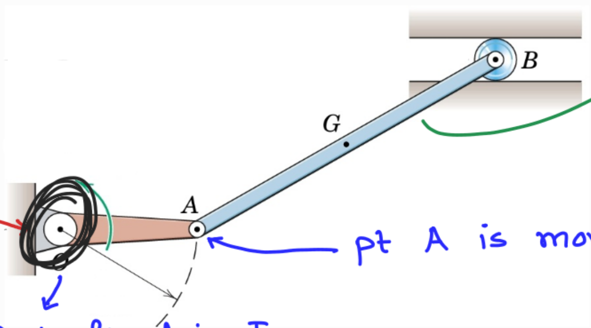
Euler's 2nd axiom for a Moving Reference point

We turn to Euler's 2nd axiom : $\boxed{\dot{\underline{H}}_{O|I} = \underline{M}_O}$

and recall that this relation holds only for a point O fixed in the inertial frame I.

How can we write Euler's 2nd axiom for rod AB about A ?

Ex:



Fixed inertial frame I

pt A is moving wrt I

O is fixed in I

Given any general point 'A' moving w.r.t. I (see above figure)

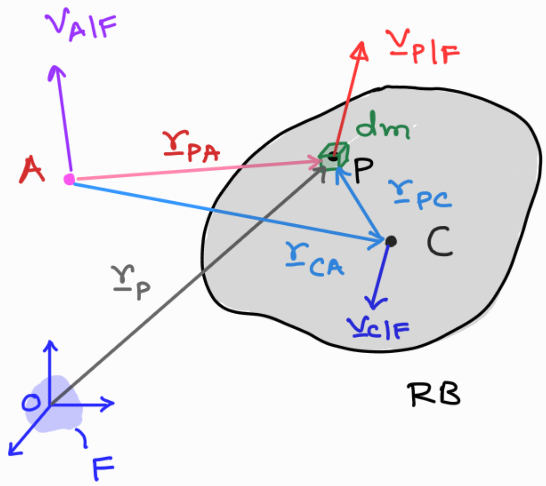
Is $\dot{\underline{\underline{H}}}_{A|I} = \underline{\underline{M}}_A$ VALID? If so, under what circumstances?

To derive an alternate form of Euler's 2nd axiom valid for a moving point and to determine any restrictions on its use, lets first derive a relationship for transfer of angular momentum from one pt to another and then we will derive a relationship between the rate of change of angular momentum about a point A and the net external moment abt A.

Transformation rule for Angular Momentum

Angular momentum of an RB abt a moving pt A was defined earlier :

$$\underline{H}_{A|F} = \int \underline{r}_{PA} \times \underline{v}_{PA|F} \, dm$$



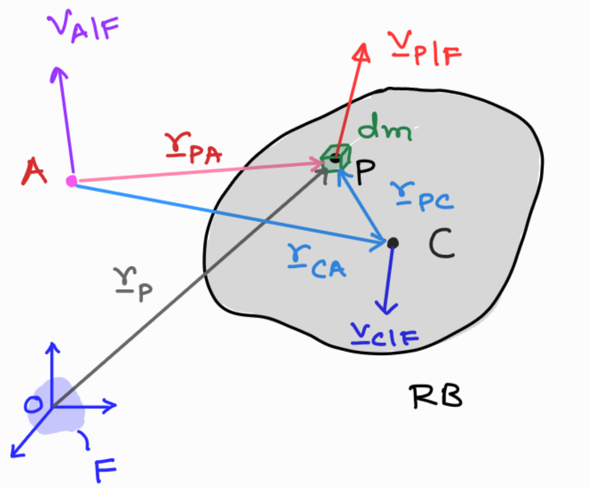
C : Center of mass of RB

It can be shown that

$$\boxed{\underline{H}_{A|F} = \underbrace{\underline{H}_{C|F}}_{\text{angular mom. abt C}} + \underbrace{m}_{\text{total mass of RB}}\left(\underline{r}_{CA} \times \underline{v}_{CA|F}\right)}$$

(Proved next)

Proof of $\underline{H}_{A|F} = \underline{H}_{C|F} + m(\underline{r}_{CA} \times \underline{v}_{CA|F})$



Start with the defn of angular momentum of RB about moving pt A

$$\underline{H}_{A|F} = \int (\underline{r}_{PA} \times \underline{v}_{PA|F})\, dm$$

$$= \int (\underline{r}_{PC} + \underline{r}_{CA}) \times \underline{v}_{PA|F}\, dm$$

$$= \int (\underline{r}_{PC} + \underline{r}_{CA}) \times (\underline{v}_{PC|F} + \underline{v}_{CA|F})\, dm$$

$$\underline{v}_{PA|F} = \underline{v}_{P|F} - \underline{v}_{A|F}$$
$$= \underline{v}_{P|F} - \underline{v}_{C|F} + \underline{v}_{C|F} - \underline{v}_{A|F}$$
$$= \underline{v}_{PC|F} + \underline{v}_{CA|F}$$

Expand

$$= \underbrace{\int (\underline{r}_{PC} \times \underline{v}_{PC|F})\, dm}_{\text{I}} + \underbrace{\int (\underline{r}_{CA} \times \underline{v}_{PC|F})\, dm}_{\text{II}}$$

$$+ \underbrace{\int (\underline{r}_{PC} \times \underline{v}_{CA|F})\, dm}_{\text{III}} + \underbrace{\int (\underline{r}_{CA} \times \underline{v}_{CA|F})\, dm}_{\text{IV}}$$

Simplify

Term I : $\int (\underline{r}_{PC} \times \underline{v}_{PC|F})\, dm = \underline{H}_{C|F}$ (angular momentum of RB abt C) by defn

Term II : $\int (\underline{r}_{CA} \times \underline{v}_{PC|F})\, dm = \underline{r}_{CA} \times \int \underline{v}_{PC|F}\, dm$

constant w.r.t. integ.

$$= \underline{r}_{CA} \times \Big\{ \underbrace{\int \underline{v}_{P|F}\, dm}_{\text{linear momentum of RB}} - \int \underline{v}_{C|F}\, dm \Big\}$$

constant

$$= \underline{r}_{CA} \times \Big\{ \underbrace{\underline{P}_{|F}}_{= m\,\underline{v}_{C|F}} - \underline{v}_{C|F} \int dm \Big\}$$

$$= \underline{r}_{CA} \times \{ m\,\underline{v}_{C|F} - m\,\underline{v}_{C|F} \}$$

$$= \underline{0}$$

Term III

$$\int (\underline{r}_{PC} \times \underline{v}_{CA|F})\, dm = \left(\int \underline{r}_{PC}\, dm\right) \times \underline{v}_{CA|F}$$

constant w.r.t the integ.

$$= \int (\underline{r}_{PO} - \underline{r}_{CO})\, dm \times \underline{v}_{CA|F}$$

$\underline{r}_{PC} = \underline{r}_{PO} - \underline{r}_{CO}$

$$= \int (\underline{r}_{P} - \underline{r}_{C})\, dm \times \underline{v}_{CA|F}$$

$$= \left(\underbrace{\int \underline{r}_{P}\, dm}_{m\,\underline{r}_C} - \underline{r}_{C} \underbrace{\int dm}_{m}\right) \times \underline{v}_{CA|F}$$

defn of COM

$$= (\cancel{m\,\underline{r}_C} - \cancel{m\,\underline{r}_C}) \times \underline{v}_{CA|F}$$

$$= \underline{0}$$

Term IV

$$\int (\underline{r}_{CA} \times \underline{v}_{CA|F})\, dm = \underbrace{(\underline{r}_{CA} \times \underline{v}_{CA|F})}_{\text{constant term w.r.t integ.}} \underbrace{\int dm}_{\text{total mass } m}$$

$$= m\,\underline{r}_{CA} \times \underline{v}_{CA|F}$$

Therefore,

$$\underline{H}_{A|F} = \underline{H}_{C|F} + m\,\underline{r}_{CA} \times \underline{v}_{CA|F}$$

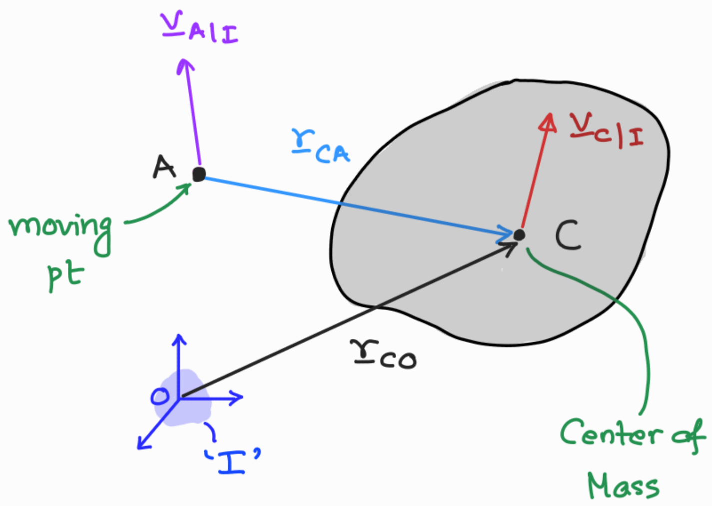
The above angular momemtum transfer rule is valid for any reference frame F, and therefore is also valid for inertial frames of reference.

Next, we intend to find a relationship between

$$\dot{\underline{H}}_{A|I} = \frac{d}{dt}\{H_{A|I}\}\Big|_I \quad \overset{??}{\longleftrightarrow} \quad \underline{M}_A$$

$\dot{\underline{H}}_{A|I} - \underline{M}_A$ relationship

- Consider now 'O' as the origin of the inertial frame 'I'
- Pt A is moving reference pt
- Pt C is the COM of RB



Using angular momentum transfer rule, we can write :

① $\underline{H}_{A|I} = \underline{H}_{C|I} + m(\underline{r}_{CA} \times \underline{v}_{CA|I})$

② $\underline{H}_{O|I} = \underline{H}_{C|I} + m(\underline{r}_{CO} \times \underline{v}_{CO|I})$

Substitute Eq. ② in Eq. ① :

$$\underline{H}_{A|I} = \left[\underline{H}_{O|I} - m(\underline{r}_{CO} \times \underline{v}_{CO|I})\right] + m(\underline{r}_{CA} \times \underline{v}_{CA|I})$$

Differentiate both sides wrt time in the 'I' frame:

$$\dot{\underline{H}}_{A|I} = \frac{d}{dt}\left\{\left[\underline{H}_{O|I} - m(\underline{r}_{CO} \times \underline{v}_{CO|I})\right] + m(\underline{r}_{CA} \times \underline{v}_{CA|I})\right\}\Big|_I$$

$$\frac{d}{dt}\left\{\underline{H}_{O|I}\right\}\Big|_I = \dot{\underline{H}}_{O|I} = \underline{M}_O \leftarrow \text{Net moment due to all ext. forces and couples about } O,\ O \in I$$

$$\frac{d}{dt}\left(\underline{r}_{CO} \times \underline{v}_{CO|I}\right)\Big|_I = \frac{d}{dt}\left\{\underline{r}_{CO}\right\}\Big|_I \times \underline{v}_{CO|I} + \underline{r}_{CO} \times \frac{d}{dt}\left\{\underline{v}_{CO|I}\right\}\Big|_I$$

$$= \underbrace{\underline{v}_{CO|I} \times \underline{v}_{CO|I}}_{\underline{0}} + \underline{r}_{CO} \times \underline{a}_{CO|I}$$

$$\frac{d}{dt}\left(\underline{r}_{CA} \times \underline{v}_{CA|I}\right)\Big|_I = \frac{d}{dt}\left\{\underline{r}_{CA}\right\}\Big|_I \times \underline{v}_{CA|I} + \underline{r}_{CA} \times \frac{d}{dt}\left\{\underline{v}_{CA|I}\right\}\Big|_I$$

$$= \underbrace{\underline{v}_{CA|I} \times \underline{v}_{CA|I}}_{\underline{0}} + \underline{r}_{CA} \times \underline{a}_{CA|I}$$

$$\dot{\underline{H}}_{A|I} = \dot{\underline{H}}_{O|I} - m\left(\underline{r}_{CO} \times \underline{a}_{CO|I}\right) + m\left(\underline{r}_{CA} \times \underline{a}_{CA|I}\right)$$

$$\underline{r}_{CO} = \underline{r}_C - \cancelto{\underline{0}}{\underline{r}_O} = \underline{r}_C, \qquad \underline{r}_{CA} = \underline{r}_C - \underline{r}_A$$

$$\underline{a}_{CO|I} = \underline{a}_{C|I} - \cancelto{\underline{0}}{\underline{a}_{O|I}} = \underline{a}_{C|I}, \qquad \underline{a}_{CA|I} = \underline{a}_{C|I} - \underline{a}_{A|I}$$

$$\dot{\underline{H}}_{A|I} = \dot{\underline{H}}_{O|I} - m\left(\underline{r}_C \times \underline{a}_{C|I}\right) + m\left[\underline{r}_{CA} \times \left(\underline{a}_{C|I} - \underline{a}_{A|I}\right)\right]$$

$$= \dot{\underline{H}}_{O|I} - m\,(\underbrace{\underline{r}_C - \underline{r}_{CA}}_{\underline{r}_A}) \times \underline{a}_{C|I} - m\,\underline{r}_{CA} \times \underline{a}_{A|I}$$

$$= \underbrace{\dot{\underline{H}}_{O|I}}_{=\underline{M}_O} - \underline{r}_A \times \underbrace{\left(m\,\underline{a}_{C|I}\right)}_{=\underline{F}_R \text{ (net force)}} - \underline{r}_{CA} \times \left(m\,\underline{a}_{A|I}\right)$$

(Euler's 2nd axiom) (Euler's 1st axiom)

$$\Rightarrow \quad \dot{\underline{H}}_{A|I} = \underline{M}_O - (\underline{r}_A \times \underline{F}_R) - (\underline{r}_{CA} \times m\,\underline{a}_{A|I})$$

Recall from Lec 8 :

$$\underline{M}_O = \underline{M}_A + \underline{r}_{AO} \times \underline{F}_R = \underline{M}_A + \underline{r}_A \times \underline{F}_R$$

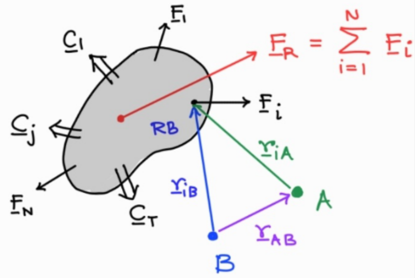
Moment abt pt B due to force system 1

$$\underline{M}_B = \sum_{i=1}^{N} \underline{r}_{iB} \times \underline{F}_i + \sum_{j=1}^{T} \underline{C}_j$$

$$= \sum_{i=1}^{N} (\underline{r}_{AB} + \underline{r}_{iA}) \times \underline{F}_i + \sum_{j=1}^{T} \underline{C}_j$$

$$= \underbrace{\sum_{i=1}^{N} \underline{r}_{iA} \times \underline{F}_i + \sum_{j=1}^{T} \underline{C}_j}_{\underline{M}_A} + \sum_{i=1}^{N} \underbrace{\underline{r}_{AB}}_{\text{same for all } i} \times \underline{F}_i$$

$$= \underline{M}_A + \underline{r}_{AB} \times \underbrace{\left(\sum_{i=1}^{N} \underline{F}_i\right)}_{\underline{F}_R} \quad \Rightarrow \quad \underline{M}_B = \underline{M}_A + \underline{r}_{AB} \times \underline{F}_R \quad \text{—} (*)$$



$$\Rightarrow \quad \dot{\underline{H}}_{A|I} = \underline{M}_A - \underline{r}_{CA} \times m\,\underline{a}_{A|I}$$

← Alternative form of Euler's 2nd axiom for moving ref. point A

Revisiting the question:

Given any general point 'A' moving w.r.t. I

Is $\dot{\underline{H}}_{A|I} = \underline{M}_A$ VALID ? If so, under what circumstances ?

$\dot{\underline{H}}_{A|I} = \underline{M}_A$ is valid when

Circumstances

$\underline{r}_{CA} \times m\,\underline{a}_{A|I} = \underline{0}$

① $\underline{r}_{CA} = \underline{0} \Rightarrow A = C \equiv \text{COM}$

$\therefore\ \dot{\underline{H}}_{C|I} = \underline{M}_C$ is ALWAYS valid about COM

OR

② $\underline{a}_{A|I} = \underline{0}$

(point 'A' is NOT accelerating in the inertial frame)

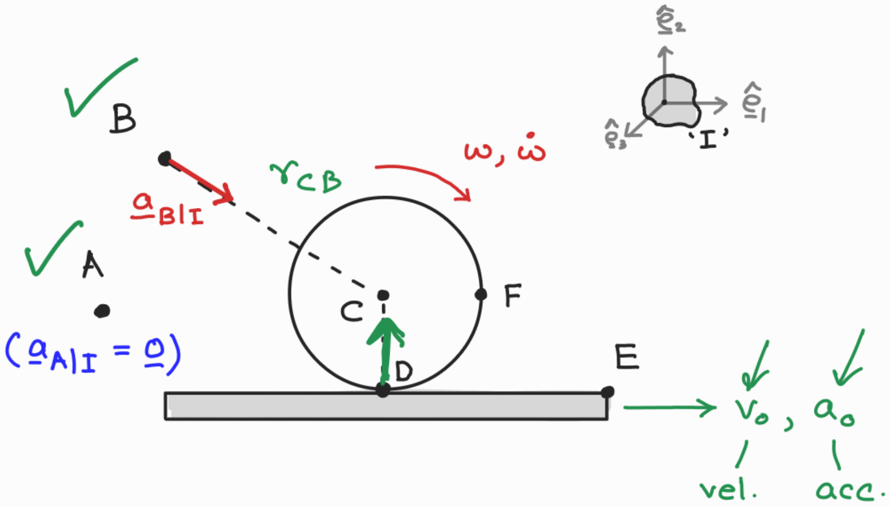
OR

③ $\underline{r}_{CA}$ // to $\underline{a}_{A|I}$

(point 'A' is accelerating towards or away from COM)

At least one of the three conditions must be met for point A for equation $\dot{\underline{H}}_{A|I} = \underline{M}_A$ to be valid for point A

Let's take an example of a uniform disc rolling without slip



$C \equiv$ Center of mass (assuming uniform mass distribution)

$D \equiv$ Contact point between plate and disc

In this example, lets list out the points 'P' for which

$$\dot{\underline{H}}_{P|I} = \underline{M}_P \text{ is valid !}$$

Points where valid

Case I : $\quad V_0 = 0$ and $a_0 = 0 \qquad$ A, B, C, D, E

Reason

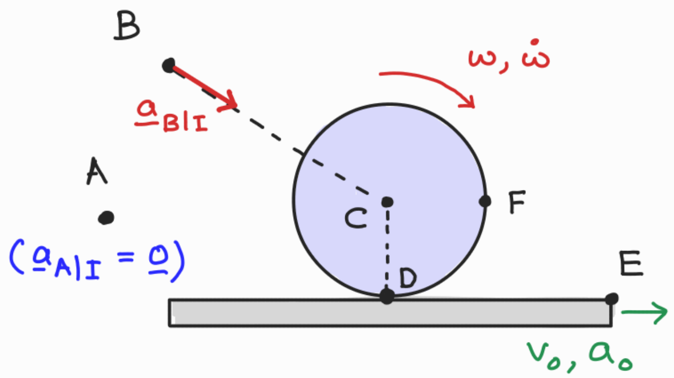
A $\rightarrow$ $\underline{a}_{A|I} = \underline{0}$

B $\rightarrow$ $\underline{r}_{CB} \,//\, \underline{a}_{B|I}$

C $\rightarrow$ $\underline{r}_{CC} = \underline{0}$ (COM)

D $\rightarrow$ $\underline{r}_{CD} \,//\, \underline{a}_{D|I} = \omega^2 r\, \hat{\underline{e}}_2$

E $\rightarrow$ $\underline{a}_{E|I} = \underline{0}$



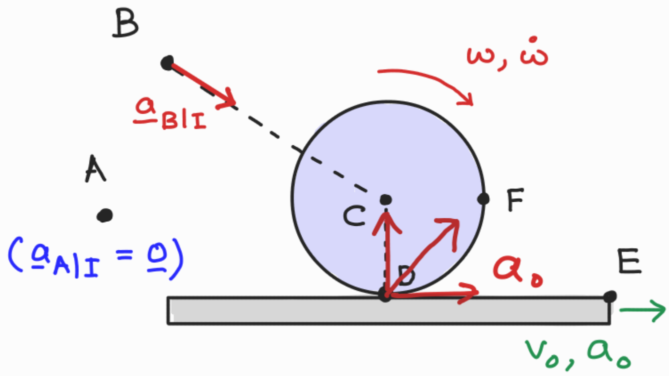
Points where valid

Case II : $\quad V_0 \neq 0$ and $a_0 = 0 \qquad$ A, B, C, D, E

Same reasons as case I

Points where valid

Case III: $V_0 \neq 0$ and $a_0 \neq 0$ $\quad$ A, B, C



Reason

A $\rightarrow$ $\underline{a}_{A|I} = \underline{0}$

B $\rightarrow$ $\underline{r}_{CB} \parallel \underline{a}_{B|I}$

C $\rightarrow$ $\underline{r}_{CC} = \underline{0}$ (COM)

D ✗ $\quad \underline{a}_{D|I} = a_0 \hat{\underline{e}}_1 + \omega^2 r \hat{\underline{e}}_2$

E ✗ $\quad \underline{a}_{E|I} = a_0 \hat{\underline{e}}_1 \neq 0$

Point F: $\quad \underline{a}_{F|I} = (-\omega^2 r + a_0)\, \hat{\underline{e}}_1 - r\dot{\omega}\, \hat{\underline{e}}_2$ (Verify)

If $\dot{\omega} = 0$, then $\underline{a}_{F|I}$ becomes $\underbrace{\text{parallel to } \underline{r}_{CF}}_{\text{accelerating towards or away from COM}}$ and pt. F becomes a valid point!

$\therefore$ pt F is valid when $\dot{\omega} = 0$