APL 100: Engineering Mechanics

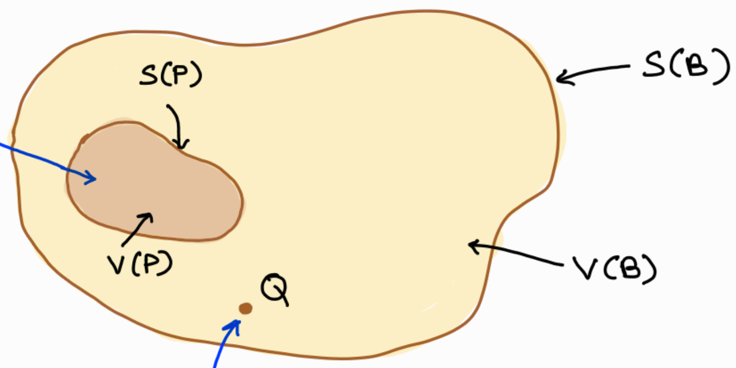
What is mechanics ?

Study of mechanical actions (forces/moments) and their relation to the state of motion (disp, vel, acc) of <u>material bodies</u>

Any object, such as electron, a ball, a river, an aircraft, a planet, is called a body.

A material body B is defined as a continuous distribution of matter (mass) of a well-defined identity whose configuration has a volume $V(B)$ of space enclosed by smooth surface $S(B)$

A part P of the body is a subset of body B which at time t occupies a volume $V(P)$ enclosed by smooth surface $S(P)$



Body B

A point Q in $V(B)$ is called a material point of the body

Matter is not continuously distributed over the volume of a body; it is composed of very large number of protons, neutrons, electrons, etc. whose arrangement is not completely known.

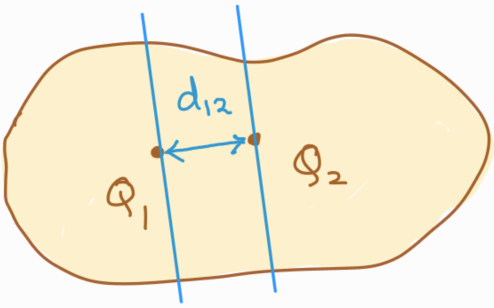
For mathematical convenience, we assume a continuous distribution of matter $\Rightarrow$ called the CONTINUUM hypothesis.

mathematical models for bodies.

- particle/point
- rigid body
- deformable body

Increasing complexity

(a) Rigid body : The distance between two material points will remain same at all times, even after forces are applied to it.



Body B

$|d_{ij}|$ = constant for all material points i and j at all times

(b) Deformable body : $|d_{ij}| \neq$ constant between material points at all times.

- Solids
- Fluids

(Not a part of this course!)

→ In this course, we will study classical mechanics using continuum hypothesis

(a) Velocity (v) $<<$ Speed of light (c) $\Rightarrow$ NOT relativistic mechanics!

(b) Size of bodies $>>$ atomic/molecular scales $\Rightarrow$ NOT quantum mechanics!

(c) We assume space is flat (Euclidean) $\Rightarrow$ Euclidean geometry is valid

Classical Mechanics

Kinematics

Study of the geometrical aspects of motion of material bodies without regard to the forces and moments that produce it

(position vector, velocity, acceleration)

Dynamics

Study of the relation between applied forces and moments and the resulting motion of material bodies

$$\sum \underline{F} = m\underline{a}$$

Statics (special case)

(when acceleration $= \underline{0}$ and $\sum \underline{F} = \underline{0}$)

Space

In Newtonian mechanics, the physical space is defined as an Euclidean 3-dimensional point space $\mathbb{R}^3$.

To locate/track a body is space, we need a reference frame and a coordinate system with an origin and 3 non-coplanar axes

Reference frame

A reference frame is a set of points in a 3D space with fixed relative distances for which Euclidean geometry is valid

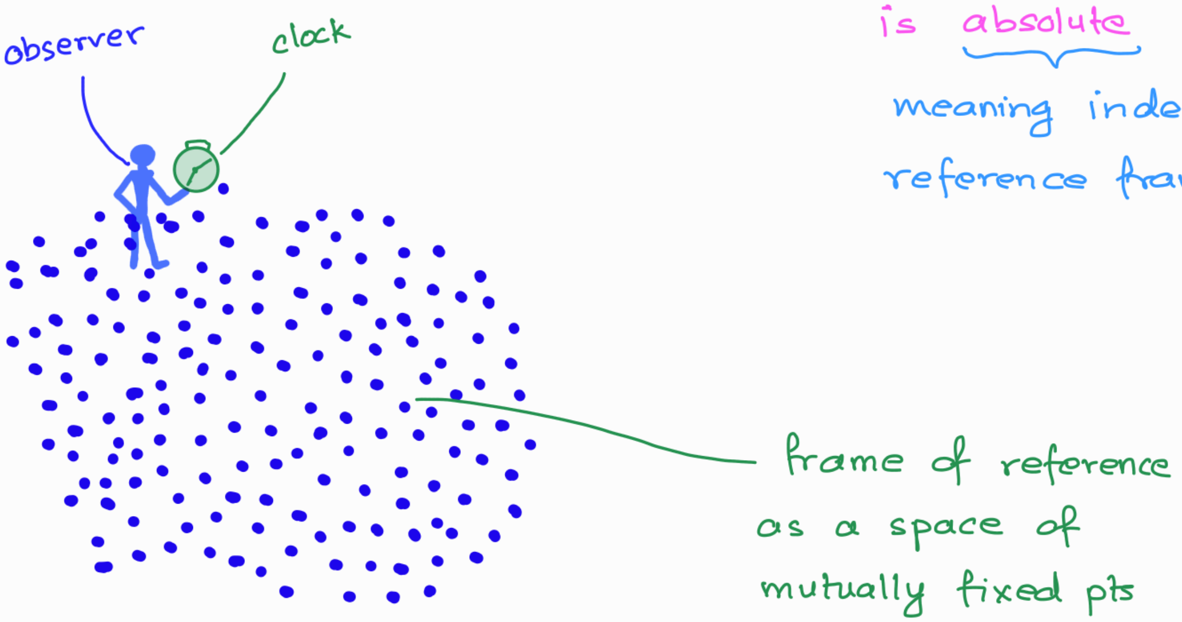
For example, the walls of a lecture class,
the cabin framework of a spacecraft
the remote stars

Reference frame has an OBSERVER with a clock

measures 'time'

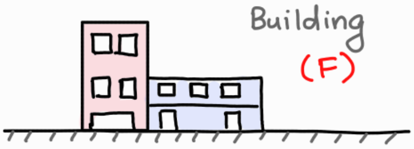
Note: Time, like any scalar, is absolute

meaning independent of reference frame

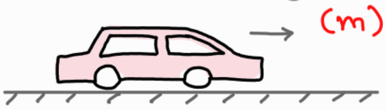


① Ref. frames may be fixed or moving in any manner

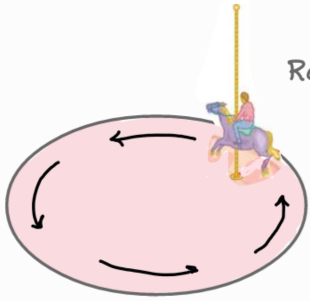
F ≡ fixed reference frame   m ≡ moving reference frame



Fixed ref. frame relative to ground



Moving ref. frame relative to ground (translation)



Moving ref. frame relative to ground (rotational)

② Ref. frames can ALWAYS be attached to $\underbrace{\text{any rigid body}}_{\text{at rest or moving}}$

$$\text{Rigid body} \xleftrightarrow[\text{in kinematics}]{\text{Synonymous}} \text{Reference frame}$$

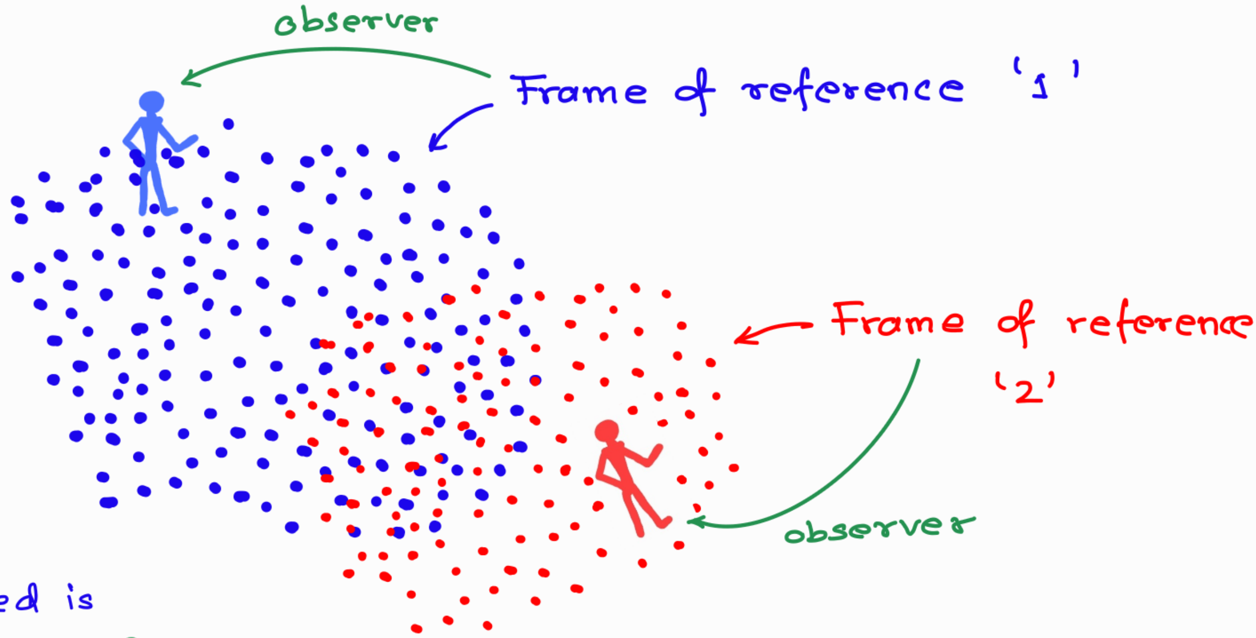
The reference frame of points fixed w.r.t a rigid body is called body-fixed-reference frame

In mechanics, choosing the "right" reference frame simplifies equations of motion.

Ex: A ground-fixed frame simplifies projectile motion problems

A frame attached to a rocket is ideal for studying its internal dynamics

③ Infinite number of ref. frames are possible



time elapsed is independent of reference frame

Coordinate System

When working with a reference frame, we need a convenient way to describe the location of points or objects in space. A coordinate system provides this structure by assigning coordinates to each point in space. These coordinates help us

a) specify positions w.r.t reference frame

b) track how objects/bodies move in relation to ref. frame

c) provides a framework to apply laws of mechanics

A coordinate system assigns an ordered set of three numbers — called coordinates — to each point in 3D space. It is defined by

(a) Origin: A point where the coordinates are zero

(b) Coordinate axes: A set of three non-coplanar unit vectors/lines

— points along these lines correspond to values of coordinates

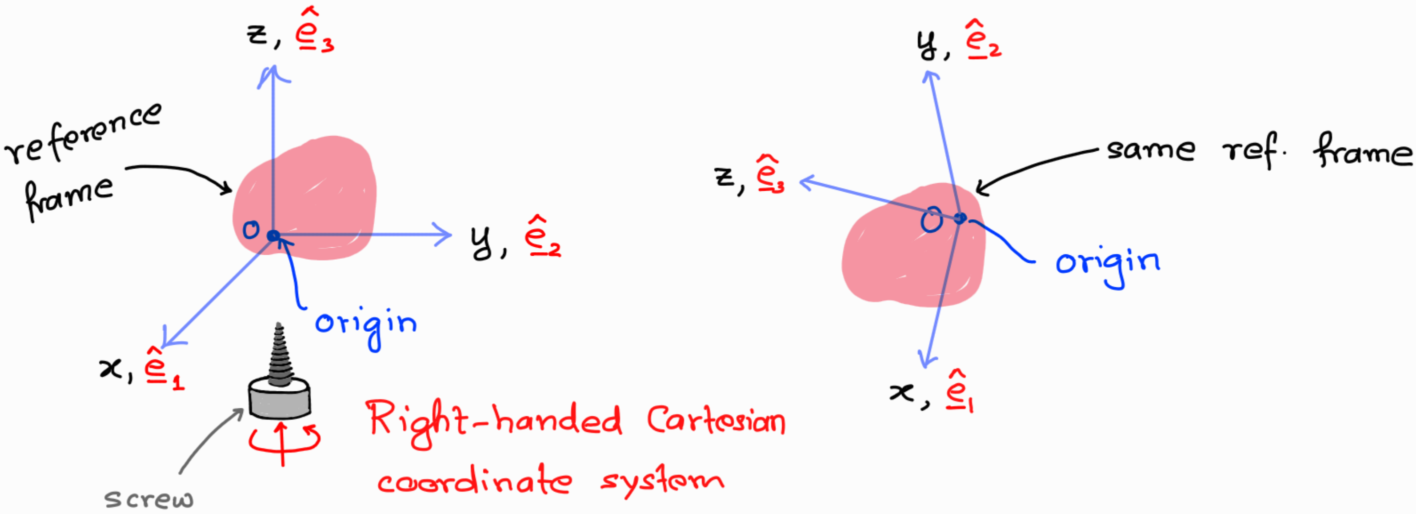
(c) If the three unit vectors are also mutually orthogonal

$\rightarrow$ csys is called orthogonal csys

Some problems are easier to solve in certain coordinate systems.

There can be <u>infinite</u> coordinate systems in a reference frame because

(a) the origin can be chosen arbitrarily within the frame

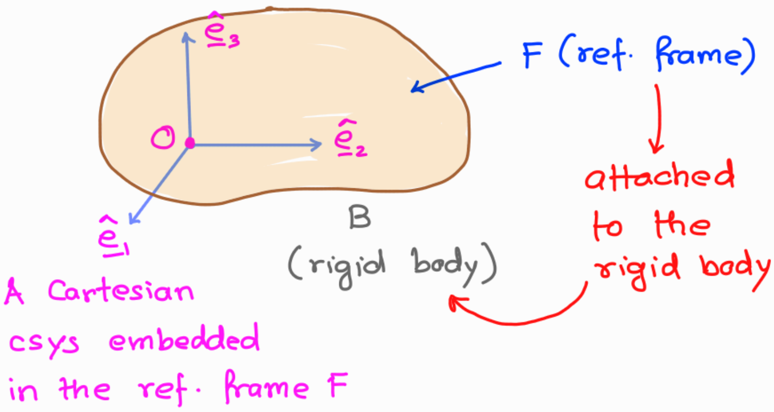


(b) the type and orientation of coordinate axes can vary (i.e. Cartesian, cylindrical)

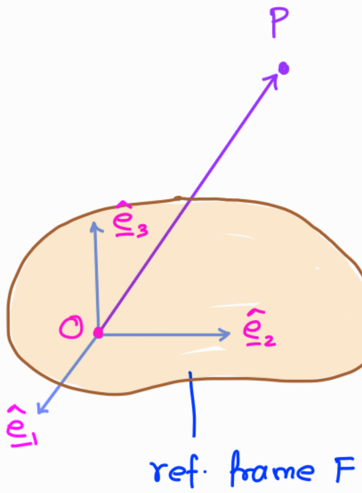
We shall use <u>underbar</u> to denote vector quantities e.g. $\underline{a}$, $\underline{e}$, $\underline{A}$

Position vector of locations

Having established a reference frame and a CSYS, as shown below:



we are now ready to define the spatial location of points from the origin O of the csys.



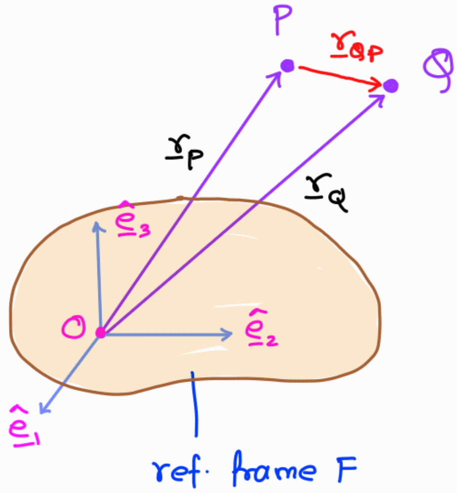
Consider a point P. We define the following:

$\overrightarrow{OP}$ ≡ directed line segment from 'O' to 'P'

$\underline{r}_{PO}$ ≡ position vector of 'P' w.r.t. 'O'

$\underline{r}_{PO} \equiv \overrightarrow{OP} \equiv \underline{r}_P$ ← Position vector of P

If position vector of a point is being shown w.r.t. the origin 'O', like in $\vec{r}_{PO}$, then we drop 'O' and simply write $\vec{r}_P$, otherwise we leave it as it is.



1. $\overrightarrow{OP} \equiv \underline{r}_{PO} = \underline{r}_P$
2. $\overrightarrow{OQ} \equiv \underline{r}_{QO} = \underline{r}_Q$
3. $\overrightarrow{PQ} \equiv \underline{r}_{QP} = \underline{r}_Q - \underline{r}_P$

$$\Rightarrow \quad \underline{r}_P + \underline{r}_{QP} = \underline{r}_Q$$

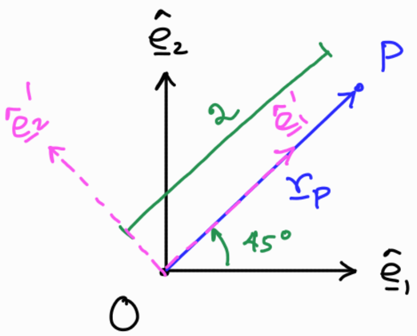
What is a vector?

(magnitude and direction)

→ It does not change if you change the reference frame

→ It is independent of the coordinate system (csys)

→ It's component relative to a csys varies from one csys to another csys!



$$\underline{r}_P = 2\cos 45^\circ\, \hat{\underline{e}}_1 + 2\sin 45^\circ\, \hat{\underline{e}}_2$$

$$[\underline{r}_P]_{(\hat{\underline{e}}_1, \hat{\underline{e}}_2)} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$\underline{r}_P = 2\, \hat{\underline{e}}_1'$$

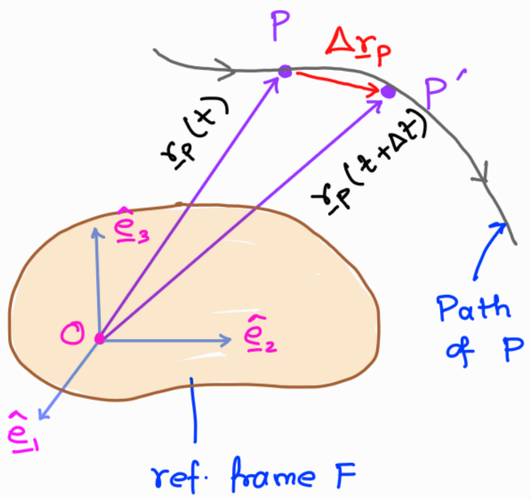
$$[\underline{r}_P]_{(\hat{\underline{e}}_1, \hat{\underline{e}}_2)} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Kinematics of a moving point w.r.t. fixed frame

A point P is in motion if it occupies different locations relative to F

The position vector of a moving pt P in ref. frame F is defined as the location occupied by it at time t relative to the origin O of csys embedded in F. It is denoted by

$$\vec{r}_P(t) \leftarrow \text{position vector is time-dependent}$$

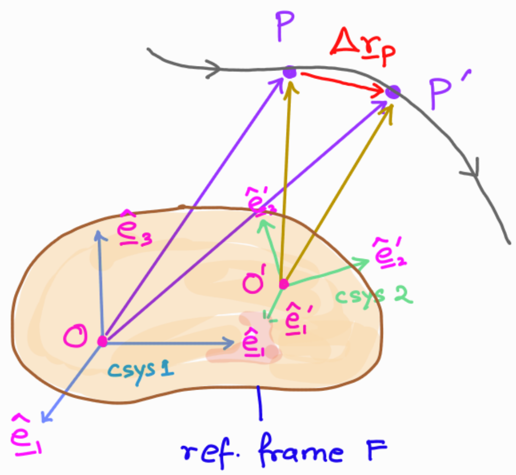


Suppose that the pt P at time t moves to another pt P′ at time t+Δt in the ref. frame F, with csys $(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$ having origin 'O' embedded in F

Then, the displacement vector of P is defined as:

$$\underline{d}_P = \Delta \underline{r}_P = \underline{r}_P(t+\Delta t) - \underline{r}_P(t)$$

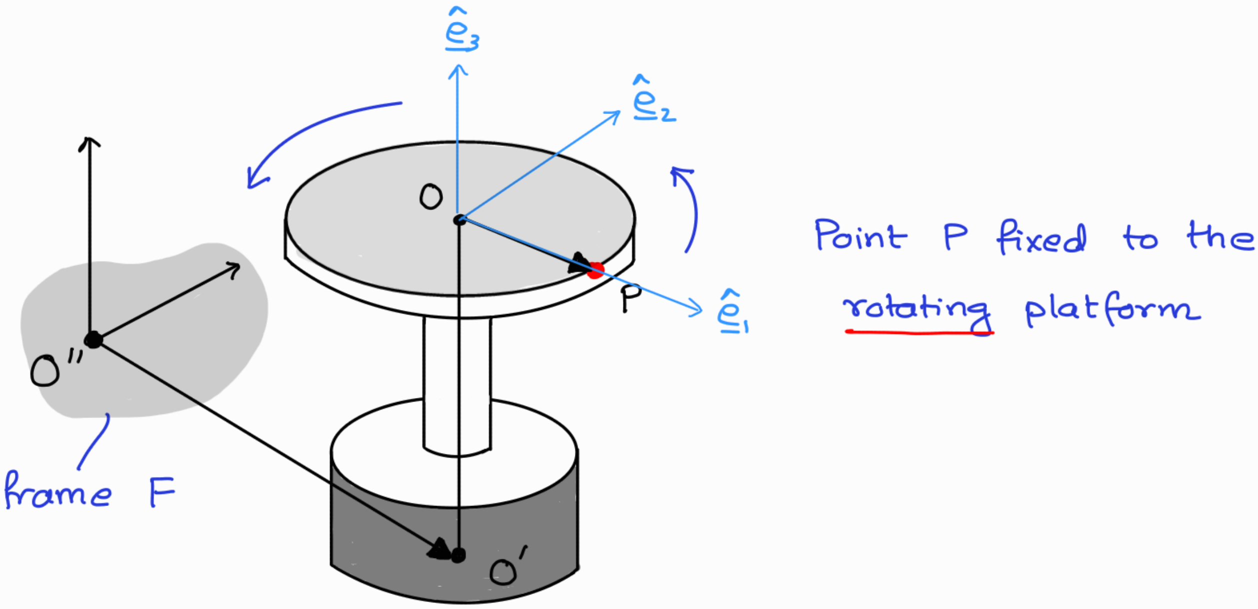
1. $\Delta \underline{r}_P$ does NOT depend on the csys origin 'O' or the orientation of the csys in the ref. frame F



Note that $\Delta \underline{r}_P = \underline{d}_P$ stays the same irrespective of the chosen csys, although the coordinate values (the representation) may be different

e.g. $\underline{d}_P = \underbrace{\frac{1}{\sqrt{2}}\hat{\underline{e}}_1 + \frac{1}{\sqrt{2}}\hat{\underline{e}}_2 + 0\,\hat{\underline{e}}_3}_{Ox_1x_2x_3 \text{ csys of F}} = \underbrace{1\,\hat{\underline{e}}_1' + 0\,\hat{\underline{e}}_2' + 0\,\hat{\underline{e}}_3'}_{Ox_1'x_2'x_3' \text{ csys of F}}$

Ex: In this example, the coordinate system could be most suitably chosen at O as that leads to straight-forward interpretation.



In time derivative, constant terms do not contribute, and this makes the origin and orientation of csys irrelevant.

If we are interested in the kinematics of point P fixed to the periphery of a platform rotating relative to ref. frame F, then it is irrelevant whether you take the origin at O, O', or O'', since:

- $\underline{r}_{PO}$ is a time-varying vector w.r.t. F
- $\underline{r}_{PO'} = \underline{r}_{PO} + \underline{r}_{OO'}$
- $\underline{r}_{PO''} = \underline{r}_{PO} + \underline{r}_{OO'} + \underline{r}_{O'O''}$

constant vectors wrt F

zero time-derivative in F

② $\Delta \underline{r}_P$ depends on the ref. frame F

Since the displacement vector $\underline{d}_P = \Delta \underline{r}_P$ does NOT depend on the csys (both origin and orientation of the axes), but depends only on the reference frame, therefore write the displacement vector explicitly w.r.t. F as: $\underline{d}_{P|F}$ ✓

and NOT $\underline{d}_P\big|_O$ ✗ or $\underline{d}_P\big|_{xyz}$ ✗

The same idea holds when defining the velocity and acceleration vectors of pt P w.r.t ref frame F

Velocity vector of point P $\equiv \underline{v}_{P|F} = \dot{\underline{r}}_P\big|_F = \left.\dfrac{d\underline{r}_P}{dt}\right|_F$

$$= \lim_{\Delta t \to 0} \left.\frac{\underline{r}_P(t+\Delta t) - \underline{r}_P(t)}{\Delta t}\right|_F$$

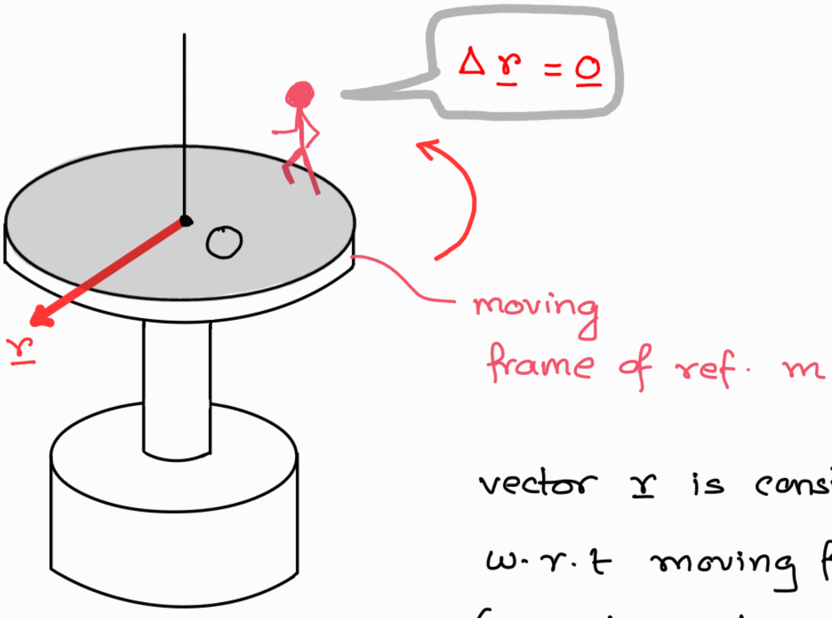
Acceleration vector of point P $\equiv \underline{a}_{P|F} = \dot{\underline{v}}_{P|F} = \left.\dfrac{d\underline{v}_P}{dt}\right|_F$

$$= \lim_{\Delta t \to 0} \left.\frac{\underline{v}_P(t+\Delta t) - \underline{v}_P(t)}{\Delta t}\right|_F$$

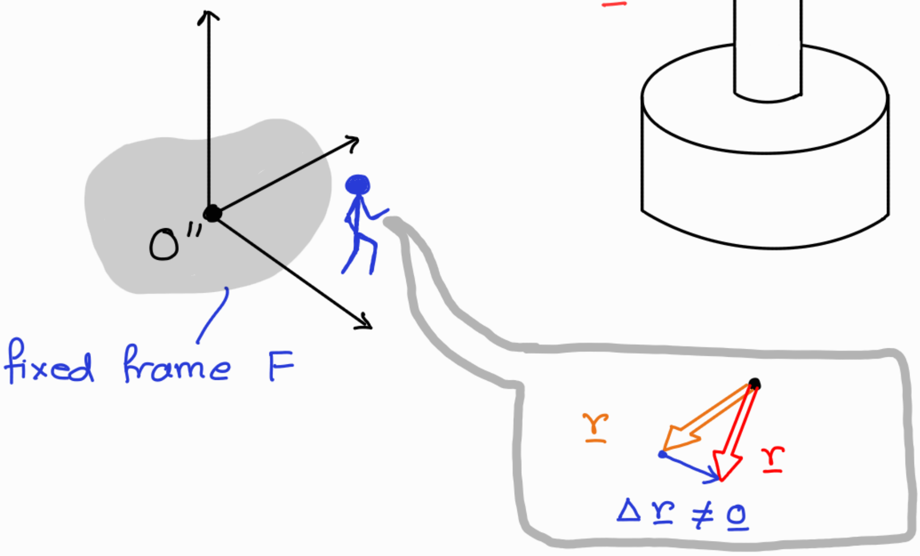
The above time-derivatives are ref. frame dependent (except when they are scalars)

For a scalar variable $r(t)$, $\dfrac{dr}{dt}$ is same for any ref. frame

However, if the variable is a vector $\underline{r}(t)$, the change of the vector $\Delta \underline{r}$ is different for different observers in diff ref. frames.

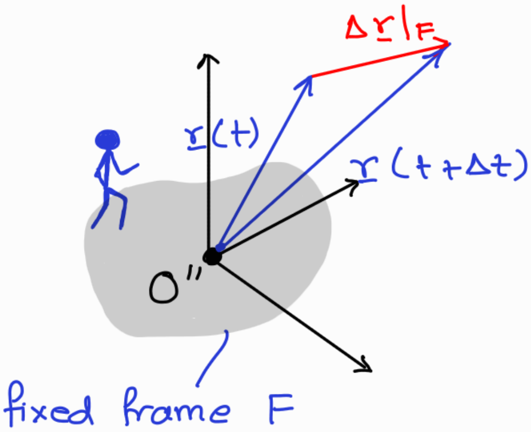


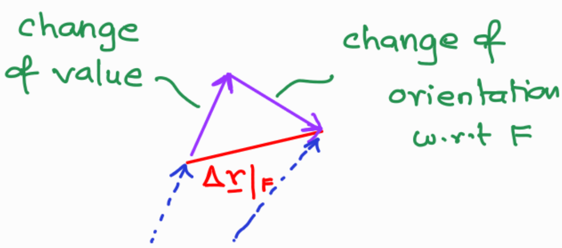
vector $\underline{r}$ is constant w.r.t moving frame (zero time-derivative)



vector $\underline{r}(t)$ changes orientation (direction) w.r.t F, hence time-derivative is not constant

Therefore, time derivative of a vector is ref. frame dependent!





The time derivative of a vector in a given reference frame F is non-zero when → there is change in value (equal in all ref. frames)

→ there is a change of orientation (which is ref. frame dependent)

→ The vector associated with the change in value is // to $\underline{r}$

→ The vector associated with the change in orientation is $\perp$ to $\underline{r}$

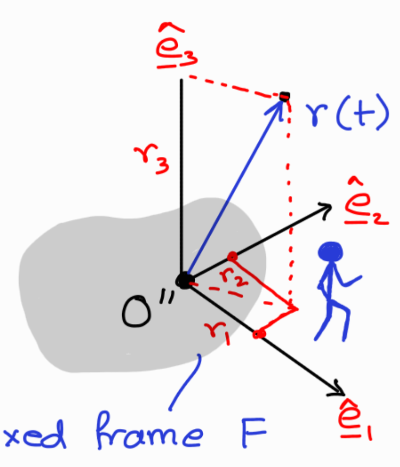
This procedure will be called geometric time derivative, and it is different from analytical time derivative which is based on the components of the vector in a given coordinate system.

Analytical time derivative of vectors

For analytical time derivative of a vector, one has to first express the vector using a csys.

For a csys attached a reference frame F, the vector $\underline{r}(t)$ may be written as:

$$\underline{r}(t) = r_1(t)\,\hat{\underline{e}}_1(t) + r_2(t)\,\hat{\underline{e}}_2(t) + r_3(t)\,\hat{\underline{e}}_3(t)$$



$$\left.\frac{d\underline{r}}{dt}\right|_F = \boxed{\frac{dr_1}{dt}}\,\hat{\underline{e}}_1 + \boxed{\frac{dr_2}{dt}}\,\hat{\underline{e}}_2 + \boxed{\frac{dr_3}{dt}}\,\hat{\underline{e}}_3$$

$$+\ r_1 \left.\frac{d\hat{\underline{e}}_1}{dt}\right|_F + r_2 \left.\frac{d\hat{\underline{e}}_2}{dt}\right|_F + r_3 \left.\frac{d\hat{\underline{e}}_3}{dt}\right|_F$$

scalar functions (does not depend on ref. frame)

vector functions depend on ref frame

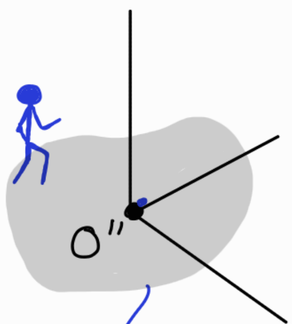
Since the csys is fixed (not moving) w.r.t. F, therefore

$$\left.\frac{d\hat{\underline{e}}_1}{dt}\right|_F = \left.\frac{d\hat{\underline{e}}_2}{dt}\right|_F = \left.\frac{d\hat{\underline{e}}_3}{dt}\right|_F = \underline{0}$$

and

$$\frac{d\underline{r}}{dt} = \frac{dr_1}{dt}\,\hat{\underline{e}}_1 + \frac{dr_2}{dt}\,\hat{\underline{e}}_2 + \frac{dr_3}{dt}\,\hat{\underline{e}}_3$$

If we consider the time derivative of the vector $\underline{r}$ projected on a moving frame of reference 'm', we will see (in Lec 4) that the time derivatives of the bases will not be zero!



fixed frame F