Index Notation

Index notation, also known as Einstein summation convention [ESC] is widely used in engineering mechanics and other fields of applied mechanics because it simplifies representations of equations, system of equations, sums, etc. involving $\underbrace{\text{scalars and vectors}}_{\text{(Tensors, in general)}}$

will be introduced later

We will learn them so that we can use them to write more compactly

Examples: ① Sum, $s = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \sum_{i=1}^{n} a_i x_i$

$$\xrightarrow{\text{ESC}} a_i x_i$$

② Systems of linear eqns:

$$\left.\begin{aligned} \hat{\underline{E}}_1 &= a_{11}\,\hat{\underline{e}}_1 + a_{12}\,\hat{\underline{e}}_2 + a_{13}\,\hat{\underline{e}}_3 \\ \hat{\underline{E}}_2 &= a_{21}\,\hat{\underline{e}}_1 + a_{22}\,\hat{\underline{e}}_2 + a_{23}\,\hat{\underline{e}}_3 \\ \hat{\underline{E}}_3 &= a_{31}\,\hat{\underline{e}}_1 + a_{32}\,\hat{\underline{e}}_2 + a_{33}\,\hat{\underline{e}}_3 \end{aligned}\right\} \xrightarrow{\text{ESC}} \hat{\underline{E}}_i = a_{ij}\,\hat{\underline{e}}_j$$

compact!

Index Notation – Definitions

① Non-repeating Index / free index:

occurs once and only once in a term

e.g. $\hat{\underline{E}}_i = a_{ij}\,\hat{\underline{e}}_j$ ← one term

$i \to$ free index

② Repeating index / dummy index / summing index:

(a) occurs twice in a term

(b) gets summed over the entire range of the index

$$\hat{E}_i = a_{ij}\,\hat{e}_j$$

Same

$j \rightarrow$ repeating index (summed over the range of the index)

(occurs twice)

$\Downarrow$

typically '3' in this course

$$\hat{E}_i = a_{ik}\,\hat{e}_k$$

Some rules in ESC :

① No index must appear more than twice

e.g. (a) $a_{ij}\,b_{ij}$ ✓ vs $a_{ij}\,b_{jj}$ ✗ ($\because$ 'j' appear thrice)

(b) $a_i\,b_i\,c_i$ ✗ ($\because$ 'i' appears thrice)

② Number of free indices must match on both sides of an equation

e.g. (a) $x_i = a_{ij}\,b_j$ ✓

$i \rightarrow$ free

$i \rightarrow$ free index

$j \rightarrow$ summing/dummy index

$\Downarrow$

$$x_1 = a_{11}\,b_1 + a_{12}\,b_2 + a_{13}\,b_3$$

$$x_2 = a_{21}\,b_1 + a_{22}\,b_2 + a_{23}\,b_3$$

(b) $x_i = a_{ij}$ ✗ (ambiguous)

(Not allowed)

'i' or 'j' both look like free indices on RHS

(c) $x_i = a_{jk} b_k + c_i$ ✗

$i \to$ free

1 free index (LHS)

$k \to$ summing dummy index

$j \to$ free index

$i \to$ free index

2 free indices (RHS)

③ Each term must have the same free indices in any valid equation

e.g. $F_i = m a_i$ ✓

$i \to$ free $\quad$ $i \to$ free

$F_i = m a_j$ ✗

$i \to$ free $\quad$ $j \to$ free

$A_{ij} = B_{ik} C_{kj}$ [Matrix multiplication] ✓

$A_{12} = B_{11} C_{12} + B_{12} C_{22} + B_{13} C_{32}$

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

two-variable function

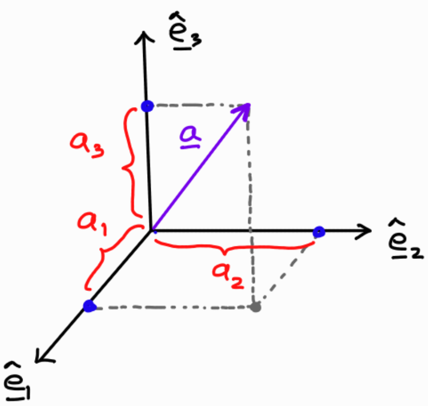
$$[\delta_{ij}] = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_{3\times3}$$

Suppose $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ form a mutually perpendicular triad of unit vectors, then:

$$\left.\begin{array}{ll} \hat{\underline{e}}_1 \cdot \hat{\underline{e}}_1 = 1, & \hat{\underline{e}}_1 \cdot \hat{\underline{e}}_2 = 0 \\ \hat{\underline{e}}_2 \cdot \hat{\underline{e}}_2 = 1, & \hat{\underline{e}}_1 \cdot \hat{\underline{e}}_3 = 0 \\ \hat{\underline{e}}_3 \cdot \hat{\underline{e}}_3 = 1, & \hat{\underline{e}}_2 \cdot \hat{\underline{e}}_3 = 0 \end{array}\right\} \Rightarrow \underbrace{\hat{\underline{e}}_i \cdot \hat{\underline{e}}_j = \delta_{ij}}_{\text{Compact}}$$

and any vector can be represented by its components using the ESC notation as:

$$\underline{a} = a_i \, \hat{\underline{e}}_i \quad \Rightarrow \quad \underline{a} = a_1 \hat{\underline{e}}_1 + a_2 \hat{\underline{e}}_2 + a_3 \hat{\underline{e}}_3$$



Similarly, $\underline{b} = b_j \, \hat{\underline{e}}_j \quad \Rightarrow \quad \underline{b} = b_1 \hat{\underline{e}}_1 + b_2 \hat{\underline{e}}_2 + b_3 \hat{\underline{e}}_3$

and, an inner product between vectors $\underline{a}$ and $\underline{b}$ in ESC will be:

vectors

$$\underline{a} \cdot \underline{b} = (a_i \, \hat{\underline{e}}_i) \cdot (b_j \, \hat{\underline{e}}_j)$$

scalars

$$= a_i \, b_j \, (\hat{\underline{e}}_i \cdot \hat{\underline{e}}_j)$$

$$= a_i \, b_j \, \delta_{ij}$$

(Here, both i and j are dummy summing indices)