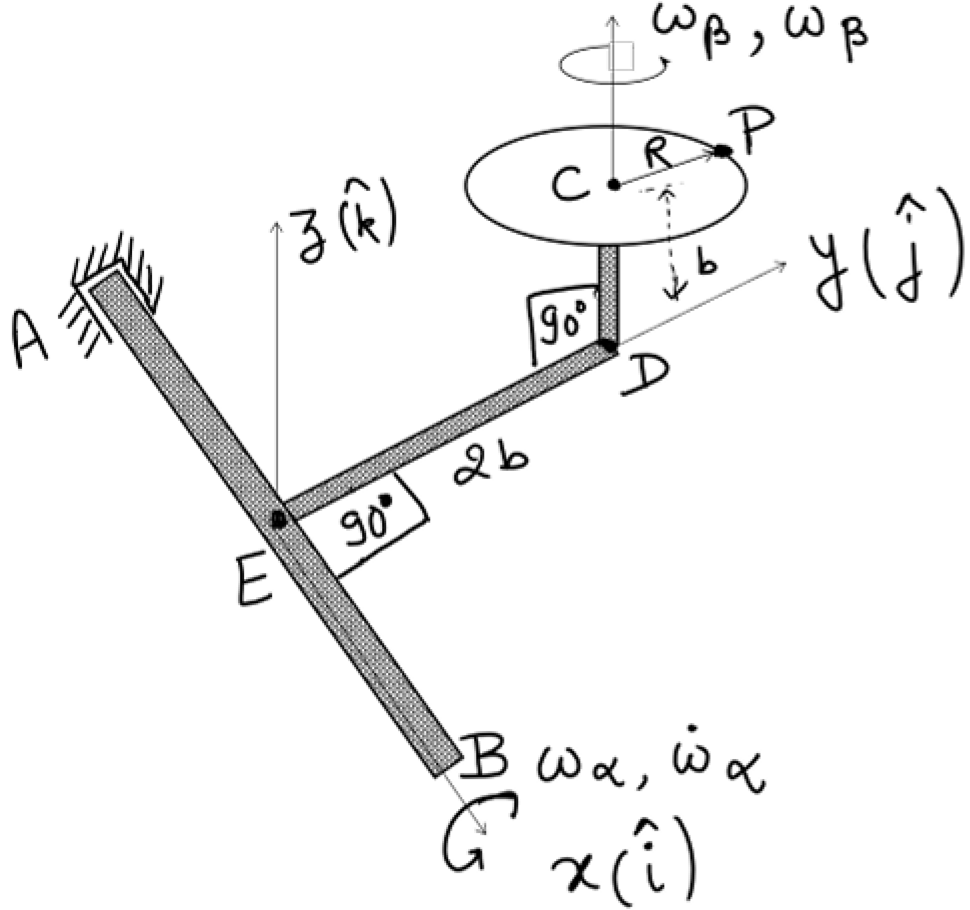


Q1) [7 points] Rods AEB and EDC are welded together (rigid body 1). A is a ball-and-socket joint. This assembly rotates relative to the ground frame with angular velocity and angular acceleration ω_α and $\dot{\omega}_\alpha$

The shown disk (rigid body 2) is connected to the rod EDC through a pin joint and is rotates with angular velocity (ω_β) and acceleration ($\dot{\omega}_\beta$) relative to the rod EDC in a direction parallel to $\hat{k}$. Find the velocity and acceleration of point P (which is fixed to the disk) with respect to ground. At this instant the line connecting C to P is along the $\hat{j}$ direction.

Please verify this solution (freshly written)

Q1:

$\underline{\omega}_{1|I} = \omega_\alpha \hat{i}$, $\dot{\underline{\omega}}_{1|I} = \dot{\omega}_\alpha \hat{i}$ (i)

$\underline{\omega}_{2|1} = \omega_\beta \hat{k}$, $\dot{\underline{\omega}}_{2|1} = \dot{\omega}_\beta \hat{k}$ (ii)

$\underline{\omega}_{2|I} = \underline{\omega}_{2|1} + \underline{\omega}_{1|I} = \omega_\beta \hat{k} + \omega_\alpha \hat{i}$ (iii)

$\dot{\underline{\omega}}_{2|I} = \dot{\underline{\omega}}_{2|1} + \dot{\underline{\omega}}_{1|I} + \underline{\omega}_{1|I} \times \underline{\omega}_{2|1}$

$= \dot{\omega}_\beta \hat{k} + \dot{\omega}_\alpha \hat{i} + \omega_\alpha \hat{i} \times \underline{\omega}_\beta \hat{k}$

$\hat{k} \circlearrowright \hat{i}, \hat{j}$

$\dot{\underline{\omega}}_{2|I} = \dot{\omega}_\beta \hat{k} + \dot{\omega}_\alpha \hat{i} + \omega_\alpha \omega_\beta (-\hat{j})$ (iv)

$\underline{r}_{PC} = R\hat{j} \rightarrow$ (v)

I.F. = ② I.P. = C (I.P ∈ 2)

$\underline{V}_{P|I} = \underline{V}_{C|I} + \underline{\omega}_{2|I} \times \underline{r}_{PC} + \cancel{\underline{V}_{P|2}}$ 0.

$\underline{V}_{C|I}$: I.F. ① I.P. = E (E ∈ ①)

$= \cancel{\underline{V}_{E|I}} + \underline{\omega}_{1|I} \times \underline{r}_{CE} + \cancel{\underline{V}_{E|1}}$

$= \omega_\alpha \hat{i} \times (2b\hat{j} + b\hat{k})$

$= 2\omega_\alpha b\hat{k} + \omega_\alpha b(-\hat{j})$

$\therefore \underline{V}_{P|I} = -b\omega_\alpha \hat{j} + 2\omega_\alpha b\hat{k} + (\omega_\beta \hat{k} + \omega_\alpha \hat{i}) \times (R\hat{j})$

$= -b\omega_\alpha \hat{j} + 2\omega_\alpha b\hat{k} + \omega_\beta R(-\hat{i}) + \omega_\alpha R(\hat{k})$

$$\boxed{\underline{V}_{P|I} = -R\omega_\beta \hat{i} + (-b\omega_\alpha)\hat{j} + (2\omega_\alpha b + \omega_\alpha R)\hat{k}}$$

I.F. = 2 , I.P. = C (∈ I.F.)

$$\underline{a}_{P|I} = \underline{a}_{C|I} + \dot{\underline{\omega}}_{2|I} \times (\underline{r}_{PC}) + \underline{\omega}_{2|I} \times (\underline{\omega}_{2|I} \times \underline{r}_{PC})$$

$$+ 2\underline{\omega}_{2|I} \times \cancel{\underline{v}_{P|2}}^{\,0} + \cancel{\underline{a}_{P|2}}^{\,0} \quad \text{(vi)} \qquad k \circlearrowleft i, \; j$$

$$\underline{\omega}_{2|I} \times (\underline{\omega}_{2|I} \times \underline{r}_{PC}) = \underline{\omega}_{2|I} \times [(\omega_\beta \hat{k} + \omega_\alpha \hat{i}) \times R\hat{j}]$$

$$= \underline{\omega}_{2|I} \times [-\omega_\beta R \hat{i} + \omega_\alpha R \hat{k}]$$

$$= (\omega_\beta \hat{k} + \omega_\alpha \hat{i}) \times (-\omega_\beta R \hat{i} + \omega_\alpha R \hat{k})$$

$$= -\omega_\beta^2 R \hat{j} - \omega_\alpha \omega_\beta (0) + \omega_\beta \omega_\alpha R (0) + \omega_\alpha^2 R (-\hat{j})$$

$$= (-\omega_\beta^2 R - \omega_\alpha^2 R)\hat{j} \quad \text{(vii)}$$

$$\dot{\underline{\omega}}_{2|I} \times \underline{r}_{PC} = (\dot{\omega}_\beta \hat{k} + \dot{\omega}_\alpha \hat{i} - \omega_\alpha \omega_\beta \hat{j}) \times R\hat{j}$$

$$= -\dot{\omega}_\beta R \hat{i} + \dot{\omega}_\alpha R \hat{k}$$

$\underline{a}_{C|I} =$ I.F. = 1 , I.P. = E (E ∈ I.F)

$$= \cancel{\underline{a}_{E|I}} + \dot{\underline{\omega}}_{1|I} \times \underline{r}_{CE} + \underline{\omega}_{1|I} \times (\underline{\omega}_{1|I} \times \underline{r}_{CE})$$

$$+ \cancel{2\underline{\omega}_{1|I} \times \underline{v}_{C|1}} + \cancel{\underline{a}_{C|1}} \quad \text{(viii)}$$

$$\underline{\omega}_{1|I} \times (\underline{\omega}_{1|I} \times \underline{r}_{CE}) = \underline{\omega}_{1|I} \times (\omega_\alpha \hat{i} \times (b\hat{k} + 2b\hat{j}))$$

$$= \underline{\omega}_{1|I} \times (-\omega_\alpha b \hat{j} + \omega_\alpha 2b \hat{k})$$

$$= \omega_\alpha \hat{i} \times (-\omega_\alpha b \hat{j} + \omega_\alpha 2b \hat{k}) \quad \text{(ix)}$$

$$= -\omega_\alpha^2 b \hat{k} - \omega_\alpha^2 2b \hat{j}$$

$$\dot{\underline{\omega}}_{1|I} \times \underline{r}_{CE} = \dot{\omega}_\alpha \hat{i} \times (b\hat{k} + 2b\hat{j})$$

$$= -\dot{\omega}_\alpha b \hat{j} + 2b\dot{\omega}_\alpha \hat{k} \quad \text{(x)}$$

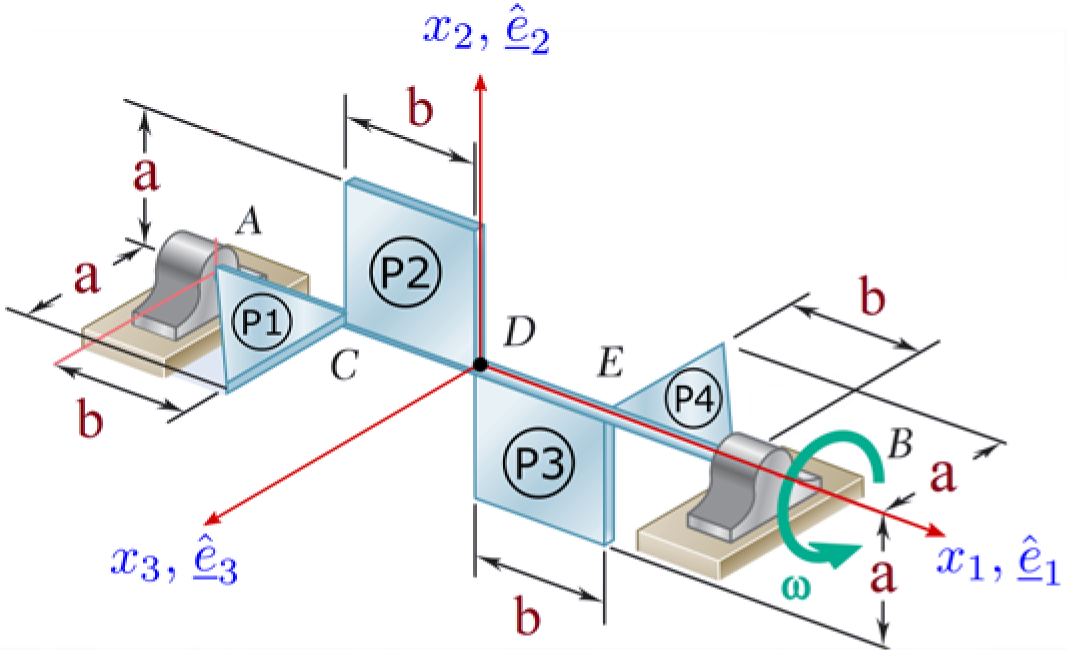
Use x, ix, viii, & vii, in vi

$$\underline{a}_{P|I} = \hat{j}(-\dot{\omega}_\alpha b - \omega_\alpha^2 2b) + \hat{k}(2b\dot{\omega}_\alpha - \omega_\alpha^2 b) - \dot{\omega}_\beta R \hat{i} + \dot{\omega}_\alpha R \hat{k}$$

$$- \omega_\beta^2 R \hat{j} - \omega_\alpha^2 R \hat{j}$$

Answer

Q2) [9 points] The shown assembly consists of two thin-rectangular plates of uniform density (mass m each) and two right triangular plates of uniform density (mass $0.5m$ each) welded to a massless rod, supported by bearings at A and B. Neglect the thickness of the plates. The assembly rotates at a constant angular velocity $\omega\underline{\hat{e}}_1$. Find the dynamic reactions at A and B using the given assembly-fixed coordinate system. The middle plates lie in the $x_1 - x_2$ plane, while the other two plates lie in the $x_1 - x_3$ plane. The bearing at A and B can generate only reaction forces but no reaction torques.



SOLUTION

Mass of sheet metal: $m = 1.25 \text{ kg}$

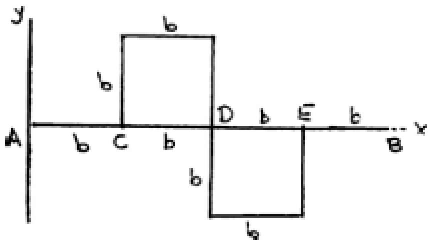
Sheet metal dimension: $b = 150 \text{ mm} = 0.15 \text{ m}$

Area of sheet metal: $A = \frac{1}{2}b^2 + b^2 + b^2 + \frac{1}{2}b^2 = 3b^2 = 0.0675 \text{ m}^2$

Let $\rho = \frac{m}{A} = \frac{1.25}{0.0675} = \frac{500}{27} \text{ kg}\cdot\text{m}^2$ = mass per unit area.

Moments and products of inertia: $I_{\text{mass}} = \rho I_{\text{area}}$

xy plane (rectangles)



$$I_x = \frac{1}{3}b^4 + \frac{1}{3}b^4 = \frac{2}{3}b^4$$

$$I_x = \frac{2}{3}\rho b^4$$

$$= \frac{2}{3}\left(\frac{500}{27}\right)(0.15)^4$$

$$= 6.25 \times 10^{-3} \text{ kg}\cdot\text{m}^2$$

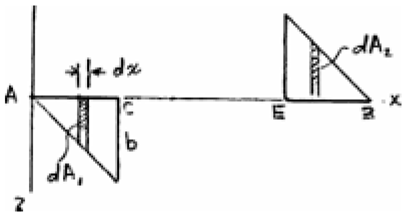
$$I_{xy} = (b^2)\left(\frac{3}{2}b\right)\left(\frac{1}{2}b\right) + (b^2)\left(\frac{5}{2}b\right)\left(-\frac{1}{2}b\right)$$

$$= -\frac{1}{2}b^4$$

$$I_{xy} = -\frac{1}{2}\rho b^4 = -\frac{1}{2}\left(\frac{500}{27}\right)(0.15)^4$$

$$= -4.6875 \times 10^{-3} \text{ kg}\cdot\text{m}^2$$

xz plane (triangles)



$$I_x = \frac{1}{12}b^4 + \frac{1}{12}b^4 = \frac{1}{6}b^4$$

$$I_x = \frac{1}{6}\rho b^4 = \frac{1}{6}\left(\frac{500}{27}\right)(0.15)^4$$

$$= 1.5625 \times 10^{-3} \text{ kg}\cdot\text{m}^2$$

PROBLEM 18.67 (Continued)

For calculation of I_{xz}, use pairs of elements dA_1 and dA_2:

$$dA_2 = dA_1.$$

$$I_{xz} = \int x\frac{z}{2}dA_1 + \int(4b-x)\left(-\frac{z}{2}\right)dA_2 = -\int(2b-x)z\,dA_1 = -\int_0^b(2b-x)z^2dx$$

but $\quad z = x.$

Hence, $\quad I_{xz} = -\int_0^a(2bx^2 - x^3)dx = -\left(\frac{2}{3}b^4 - \frac{1}{4}b^4\right) = -\frac{5}{12}b^4$

$$I_{xz} = -\frac{5}{12}\rho b^4 = -\left(\frac{5}{12}\right)\left(\frac{500}{27}\right)(0.15)^4 = -3.90625\times10^{-3}\ \text{kg}\cdot\text{m}^2$$

Total for I_x: $\quad I_x = 6.25\times10^{-3} + 1.5625\times10^{-3} = 7.8125\times10^{-3}\ \text{kg}\cdot\text{m}^2$

The mass center lies on the rotation axis, therefore

$$\bar{\mathbf{a}} = 0$$

$$\Sigma\mathbf{F} = \mathbf{A} + \mathbf{B} = m\bar{\mathbf{a}} = 0 \quad \mathbf{A} = -\mathbf{B}$$

$$\mathbf{H}_A = I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k} \quad \boldsymbol{\omega} = \omega\mathbf{i}, \quad \boldsymbol{\alpha} = \alpha\mathbf{i}$$

Let the frame of reference *Axyz* be rotating with angular velocity

$$\boldsymbol{\Omega} = \boldsymbol{\omega} = \omega\mathbf{i}$$

$$\Sigma\mathbf{M}_A = \dot{\mathbf{H}}_A = (\dot{\mathbf{H}}_A)_{Axyz} + \boldsymbol{\Omega}\times\mathbf{H}_A$$

$$M_0\mathbf{i} + 4b\mathbf{i}\times(B_y\mathbf{j} + B_z\mathbf{k}) = I_x\alpha\mathbf{i} - I_{xy}\alpha\mathbf{j} - I_{xz}\alpha\mathbf{k} + \omega\mathbf{i}\times(I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k})$$

$$M_0\mathbf{i} - 4bB_z\mathbf{j} + 4bB_y\mathbf{k} = I_x\alpha\mathbf{i} - (I_{xy}\alpha - I_{xz}\omega^2)\mathbf{j} - (I_{xz}\alpha + I_{xy}\omega^2)\mathbf{k}$$

Resolve into components and solve for B_y and B_z.

$$\mathbf{i}: \quad M_0 = I_x\alpha$$

$$\mathbf{j}: \quad B_z = \frac{(I_{xy}\alpha - I_{xz}\omega^2)}{4b}$$

$$\mathbf{k}: \quad B_y = -\frac{(I_{xz}\alpha + I_{xy}\omega^2)}{4b}$$

Data: $\quad \alpha = 0, \quad \omega = \frac{2\pi(240)}{60} = 25.133\ \text{rad/s}, \quad b = 0.15\ \text{m} \quad M_0 = 0$

$$B_z = \frac{0 - (-3.90625\times10^{-3})(25.133)^2}{(4)(0.15)} = 4.1124\ \text{N}$$

PROBLEM 18.67 (Continued)

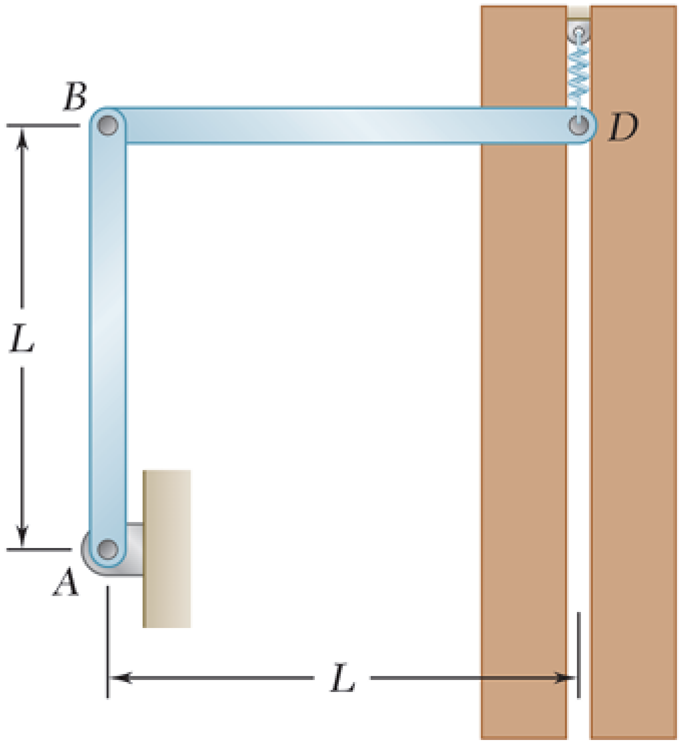
$$B_y = \frac{0 + (-4.6875 \times 10^{-3})(25.133)^2}{(4)(0.15)} = 4.9349 \text{ N}$$

$$A_y = -B_y = -4.9349 \text{ N}$$

$$A_z = -B_z = -4.1124 \text{ N}$$

$$\mathbf{A} = -(4.93 \text{ N})\mathbf{j} - (4.11 \text{ N})\mathbf{k} \blacktriangleleft$$

$$\mathbf{B} = (4.93 \text{ N})\mathbf{j} + (4.11 \text{ N})\mathbf{k} \blacktriangleleft$$



Q3) [6 points] Each of the two rods shown is of length $L = 1$ m and has a mass of 5 kg. Point D is connected to a spring of constant $k = 20$ N/m and is constrained to move along a vertical slot. Knowing that the system is released from rest when rod BD is horizontal and the spring connected to point D is initially unstretched, determine the velocity of point D when it is directly to the right of point A.

SOLUTION

Moments of inertia. $\bar{I} = \frac{1}{12} mL^2, \quad I_A = \frac{1}{3} mL^2$

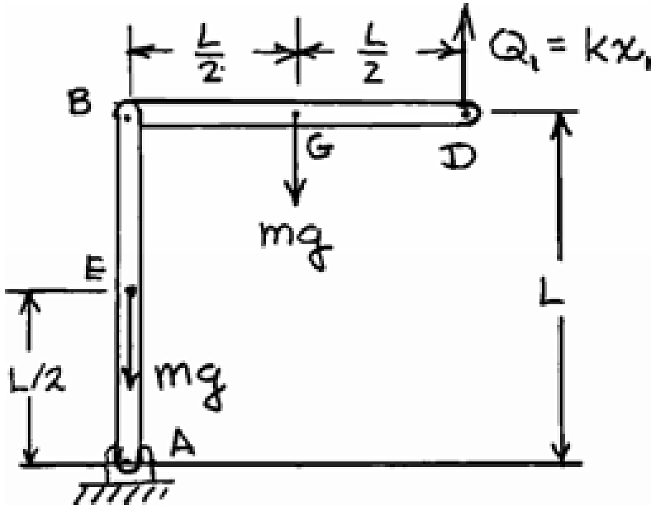
Use the principle of conservation of energy applied to the system consisting of both rods. Use the level at A as the datum for the potential energy of each rod.

Position 1. (no motion)

$$T_1 = 0$$

$$V_1 = mg\left(\frac{1}{2}L\right) + mgL + \frac{1}{2}kx_1^2$$

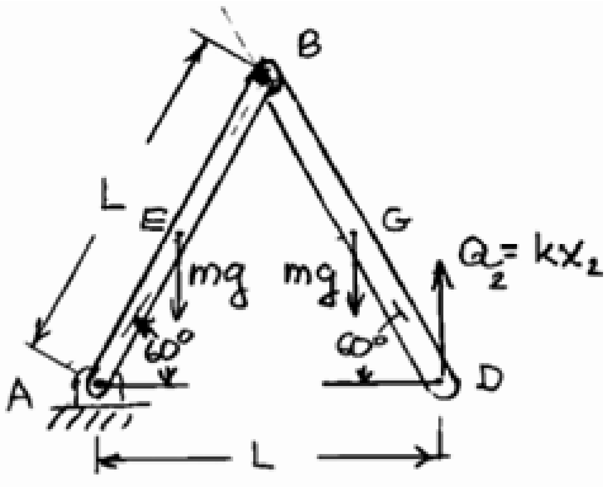
$$= \frac{3}{2}mgL + \frac{1}{2}kx_1^2$$



Position 2.

$$V_2 = mg\frac{L}{2}\sin 60° + mg\frac{L}{2}\sin 60°$$

$$= \frac{\sqrt{3}}{2}mgL + \frac{1}{2}kx_2^2$$

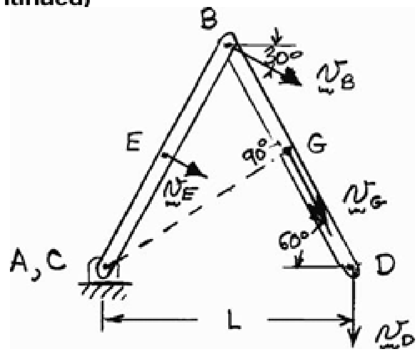


PROBLEM 17.42 (Continued)

Kinematics. $\boldsymbol{\omega}_{AB} = \omega_{AB} \circlearrowright$

$v_B = L\omega_{AB} \quad \mathbf{v}_B = L\omega_{AB} \measuredangle 30°$

$\mathbf{v}_D = v_D \downarrow$



Locate the instantaneous center C of rod BD by drawing BC perpendicular to $\mathbf{v}_B$ and DC perpendicular to $\mathbf{v}_D$. Point C coincides with Point A in position 2.

Let $\boldsymbol{\omega}_{BD} = \omega_{BD} \circlearrowright$

$$\omega_{BD} = \frac{v_B}{L} = \omega_{AB}$$

$$v_E = \frac{L}{2}\omega_{AB}$$

$$v_G = (L\sin 60°)\omega_{BD} = \frac{\sqrt{3}}{2}L\omega_{AB}$$

$$v_D = L\omega_{BD} = L\omega_{AB} \tag{1}$$

$$T_2 = \frac{1}{2}I_A\omega_{AB}^2 + \frac{1}{2}\bar{I}\omega_{BD}^2 + \frac{1}{2}mv_G^2$$

$$= \frac{1}{2}\left(\frac{1}{3}mL^2\right)\omega_{AB}^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\omega_{AB}^2 + \frac{1}{2}m\left(\frac{\sqrt{3}}{2}\omega_{AB}\right)^2$$

$$= \left(\frac{1}{6} + \frac{1}{24} + \frac{3}{8}\right)mL^2\omega_{AB}^2 = \frac{7}{12}mL^2\omega_{AB}^2$$

Principle of conservation of energy.

$$T_1 + V_1 = T_2 + V_2: \quad 0 + \frac{3}{2}mgL + \frac{1}{2}kx_1^2 = \frac{7}{12}mL^2\omega_{AB}^2 + \frac{\sqrt{3}}{2}mgL + \frac{1}{2}kx_2^2$$

$$\frac{7}{12}mL^2\omega_{AB}^2 = \left(\frac{3}{2} - \frac{\sqrt{3}}{2}\right)mgL - \frac{1}{2}k(x_2^2 - x_1^2) \tag{2}$$

Data: $m = 5$ kg, $L = 1$ m, $g = 9.81$ m/s^2

$k = 20$ N $\cdot$ m, $x_1 = 0$, $x_2 = L = 1$ m

$$\left(\frac{3}{2} - \frac{\sqrt{3}}{2}\right)mgL = (0.63397)(5\text{ kg})(9.81\text{ m/s}^2)(1\text{ m}) = 31.096\text{ J}$$

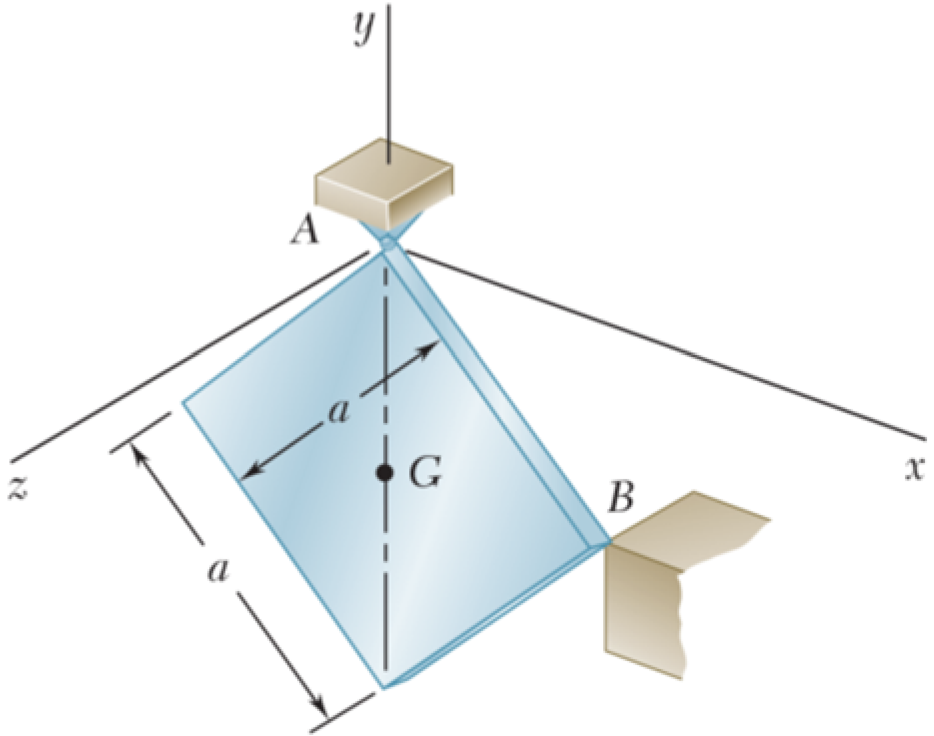
$$-\frac{1}{2}k(x_2^2 - x_1^2) = \frac{1}{2}(20\text{ N/m})(1\text{ m})^2 = -10\text{ J}$$

PROBLEM 17.42 (Continued)

By Eq. (2), $$\frac{7}{12}mL^2\omega_{AB}^2 = \left(\frac{35}{12}\text{kg}\cdot\text{m}^2\right)\omega_{AB}^2 = 21.096\text{ J}$$

$$\omega_{AB}^2 = 7.2329\text{ rad}^2/\text{s}^2 \quad \omega_{AB} = 2.6894\text{ rad/s}$$

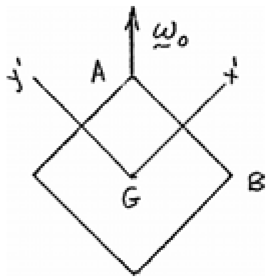
By Eq. (1), $v_D = (1\text{ m})(2.6894\text{ rad/s})$ $\mathbf{v}_D = 2.69\text{ m/s}\downarrow$ ◀



Q4) [8 points] A square plate of side a and mass m supported by a ball-and-socket joint at A is rotating about the y-axis with a constant angular velocity $\underline{\omega} = \omega_0 \underline{j}$ when an obstruction is suddenly introduced at B in the $x-y$ plane. Assuming the impact at B to be perfectly plastic ($e = 0$), determine immediately after impact (a) the angular velocity of the plate, (b) the velocity of its center of mass G.

SOLUTION

For the x' and y' axes shown, the initial angular velocity $\omega_0 \mathbf{j}$ has components



$$\omega_{x'} = \frac{\sqrt{2}}{2}\omega_0, \qquad \omega_{y'} = \frac{\sqrt{2}}{2}\omega_0,$$

Initial angular momentum about the mass center:

$$(\mathbf{H}_G)_0 = \bar{I}_{x'}\omega_{x'}\mathbf{i}' + \bar{I}_{y'}\omega_{y'}\mathbf{j}' = \frac{1}{12}ma^2\frac{\sqrt{2}}{2}\omega_0(\mathbf{i}' + \mathbf{j}')$$

Initial velocity of the mass center: $\quad \bar{\mathbf{v}}_0 = 0$

Let $\boldsymbol{\omega}$ be the angular velocity and $\bar{\mathbf{v}}$ be the velocity of the mass center immediately after impact.
Let $(F\Delta t)\mathbf{k}$ be the impulse at B.

Kinematics: $\quad \mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/A} = (\omega_{x'}\mathbf{i}' + \omega_{y'}\mathbf{j}' + \omega_{z'}\mathbf{k}') \times (-a\mathbf{j}')$

$$\mathbf{v}_B = a(\omega_{z'}\mathbf{i}' + \omega_{x'}\mathbf{k}')$$

Since the corner B does not rebound, $\quad (v_B)_{z'} = 0 \quad$ or $\quad \omega_{x'} = 0$

$$\bar{\mathbf{v}} = \boldsymbol{\omega} \times \mathbf{r}_{G/A} = (\omega_{y'}\mathbf{j}' + \omega_{z'}\mathbf{k}') \times \left(\frac{1}{2}a\right)(-\mathbf{i}' - \mathbf{j}')$$

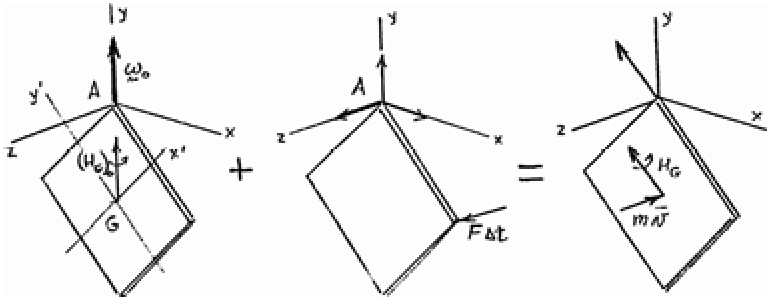
$$= \frac{1}{2}a(\omega_{z'}\mathbf{i}' - \omega_{z'}\mathbf{j} + \omega_{y}\mathbf{k}')$$

Also, $\quad \mathbf{r}_{G/A} \times m\bar{\mathbf{v}} = \frac{1}{4}ma^2(-\omega_{y'}\mathbf{i}' + \omega_{y'}\mathbf{j}' + 2\omega_{z'}\mathbf{k}')$

and $\quad \mathbf{H}_G = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}' + \bar{I}_{z'}\omega_{z'}\mathbf{k}' = \frac{1}{12}ma^2\omega_{y'}\mathbf{j}' + \frac{1}{6}ma^2\omega_{z'}\mathbf{k}'$

PROBLEM 18.31 (Continued)

Principle of impulse-momentum.



Moments about A:

$$(\mathbf{H}_A)_0 + (-a\mathbf{j}) \times (F\Delta t)\mathbf{k} = \mathbf{H}_A$$

$$(\mathbf{H}_G)_0 + \mathbf{r}_{G/A} \times m\overline{\mathbf{v}}_0 - (aF\Delta t)\mathbf{i} = \mathbf{H}_G + \mathbf{r}_{G/A} \times m\overline{\mathbf{v}}$$

Resolve into components.

$$\mathbf{i}': \quad \frac{1}{24}\sqrt{2}ma^2\omega_0 - aF(\Delta t) = -\frac{1}{4}ma^2\omega_{y'}$$

$$\mathbf{j}': \quad \frac{1}{24}\sqrt{2}ma^2\omega_0 = \frac{1}{12}ma^2\omega_{y'} + \frac{1}{4}ma^2\omega_{y'} \qquad \omega_{y'} = \frac{\sqrt{2}}{8}\omega_0$$

$$\mathbf{k}': \quad 0 = \frac{1}{6}ma^2\omega_{z'} + \frac{1}{2}ma^2\omega_{z'} \qquad \omega_{z'} = 0$$

(*a*) $$\boldsymbol{\omega} = \frac{\sqrt{2}}{8}\omega_0\mathbf{j}' = \frac{1}{8}\sqrt{2}\omega_0\frac{\sqrt{2}}{2}(\mathbf{j} - \mathbf{i}) \qquad \boldsymbol{\omega} = \frac{1}{8}\omega_0(-\mathbf{i} + \mathbf{j}) \blacktriangleleft$$

(*b*) $$\overline{\mathbf{v}} = \frac{1}{2}a\omega_y\mathbf{k}' = \frac{\sqrt{2}}{16}a\omega_0\mathbf{k} \qquad \overline{\mathbf{v}} = 0.0884a\omega_0\mathbf{k} \blacktriangleleft$$